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Ergodicity and Metastability for the Stochastic Quantisation Equation

by

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Contents

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Declarations

I hereby declare that the material in this thesis is original and a product of my own research, unless otherwise stated, conducted under the supervision of Professor Hendrik Weber during my PhD training at the University of Warwick.

Parts of this thesis have already appeared in the two articles [TW18b] and [TW18a]. Notably,

I. Chapters 2, 3 and 4 have appeared in [TW18b], published in "*Annales de l'Institut Henri Poincaré Probabilités et Statistiques*, 54(3):1204–1249, 2018".

II. Chapter 5 has appeared in the preprint "*arXiv: 1808.04171*" [TW18a], submitted for publication.

This thesis is submitted to the University of Warwick for the degree of *Doctor of Philosophy* and I confirm that it has not been submitted to any other university or for any other degree.

Abstract

In this thesis we study ergodicity and metastability of solutions to the stochastic quantisation equation of the $\mathcal{P}(\varphi)_2$-Euclidean Quantum Field theory. The main difficulty arises from the fact that solutions of this equation can be interpreted only in a renormalised sense and classical methods from SPDE Theory do not apply in this case.

I. Ergodicity:

In this part we study the long time behaviour of the law of the solutions. We first prove three main results: A strong dissipative bound for the solutions uniformly in the initial condition, the strong Feller property (and in particular local Hölder continuity of the associated Markov semigroup) and a support theorem. As a corollary, we prove exponential mixing of the law of the solutions with respect to the total variation distance.

II. Metastability:

In this part we restrict ourselves to the special case of the 2-dimensional Allen–Cahn equation perturbed by small noise and study the long time behaviour of solutions path-wisely. We prove that solutions that start close to the minimisers of the potential of the deterministic system contract exponentially fast with overwhelming probability. The exponential rate is explicit in the parameters of the equation. As an application, we prove an Eyring–Kramers law for the transition times of the solutions between the minimisers of the potential of the deterministic system.

Μὴ εἶναι βασιλικὴν ἀτραπὸν ἐπὶ γεωμετρίαν.

Εὐκλείδης

Chapter 1

Introduction

Stochastic partial differential equations (SPDEs) describe the time evolution of quantities that are influenced by a stochastic forcing. A typical example of such a forcing is space-time white noise, a Gaussian random (Schwartz) distribution which is irregular both in time and space. It is a generalisation of the derivative of the Brownian motion to higher dimensions. When the stochastic forcing is irregular, as in the case of space-time white noise, the solutions of these equations are expected to have low regularity. In many interesting cases, for example when the equation involves non-linear terms, the expected low regularity of solutions makes the system singular, that is, many terms appearing in the equation are ill-defined and classical definitions fail to provide a meaningful solution; we refer to these cases as *singular SPDEs*.

One important reason to study singular SPDEs is that they appear naturally in various fields. For example, they describe the natural reversible dynamics of “non-trivial” infinite dimensional measures in the context of constructive Quantum Field Theory (see for example [PW81]). They also arise as scaling limits of discrete models in Statistical Mechanics (see for example [GLP99]). However, the fact that singular SPDEs involve terms that are ill-defined is a major obstacle in their study and without a meaningful solution it is impossible to investigate properties of the underlying models.

Recently, Hairer’s pioneering work on *Regularity Structures* [Hai14] gave a rigorous construction of solutions for a wide variety of singular SPDEs locally in time. Around the same time, the theory of *Paracontrolled Calculus* [GIP15], developed by Gubinnelli, Imkeller and Perkowski, suggested an alternative approach for local well-posedness which, although less general, allowed the treatment of many interesting examples of singular SPDEs. These developments triggered a lot of research activity and in the last few years there has been a rapid progress in the field.

1.1 The Dynamic Φ^4_d

There are many interesting examples of singular SPDEs but describing each one of them is beyond our reach. As our main example in this introduction, let us consider the dynamic Φ^4_d which is one of the core models of our work.

The dynamic Φ^4_d is formally given by the equation

$$(\partial_t - \Delta)X = -X^3 + \mathfrak{m}X + \sqrt{2}\xi, \tag{1.1}$$

where ξ is a space-time white noise, the space variable is d-dimensional and $\mathfrak{m}$ is a real parameter denoting the mass of the system. When $d = 1$ (1.1) is well-posed. It can be treated as a classical PDE problem and has been studied extensively by many authors (see for example [Iwa87a, Iwa87b, Zab89] and the references therein). However, when $d \geq 2$ the equation is singular and classical solution theory for (S)PDEs fails to provide any meaningful solution.

Equation (1.1) was first proposed by Parisi and Wu [PW81] as the natural reversible dynamics of the Φ^4_d-Euclidean Quantum Field theory, which is formally described by the infinite dimensional measure

$$\nu(\mathrm{d}X) \propto \exp\left\{-\int\left(\frac{1}{2}|\nabla X(z)|^2 + \frac{1}{4}X(z)^4 - \frac{\mathfrak{m}}{2}X(z)^2\right)\mathrm{d}z\right\}\prod_z \mathrm{d}X(z). \tag{1.2}$$

The construction of (a renormalised version of) (1.2) in $d = 2$ and 3 was a major result in the programme of constructive Quantum Field Theory in the late 60s and 70s (see for example [Gli68, GJ73, Nel73, Fel74, GRS75, BFS83]; we also refer the reader to [GJ87] and the references therein). Parisi and Wu's original proposal, also known as *stochastic quantisation*, was to construct and study the measure (1.2) as the *equilibrium* limit of the solutions to (1.1) as $t \to \infty$ or, in other words, study *ergodicity* of solutions.

Parisi and Wu's proposal to construct the dynamics of (1.2) appealed many researchers, with several important contributions over the years. Early attempts concerned the construction in dimension $d = 2$. In [JLM85], Jona–Lasinio and Mitter used Girsanov's transformation to solve a modified equation given by

$$\left(\partial_t - (-\Delta + 1)^{-\varepsilon}\Delta\right) X = (-\Delta + 1)^{-\varepsilon}\left(-X^3 + \mathfrak{m}X\right) + \sqrt{2}(-\Delta + 1)^{-\frac{\varepsilon}{2}}\xi \quad (1.3)$$

for a class of strictly positive values of ε. Notice that formally the measure (1.2) is also reversible with respect to (1.3). Then, in [AR91], Albeverio and Röckner constructed weak solutions to (1.1) using the theory of Dirichlet forms. A stronger result was obtained in the celebrated work [DPD03] by Da Prato and Debussche. They proposed a simple transformation of (1.1) which allowed them to prove existence of strong solutions locally in time for any initial condition of suitable regularity and non-explosion for initial conditions in a set of measure one with respect to the formal equilibrium (1.2).

Although the dynamic Φ^4_d received a lot of attention as a "toy model" in constructive Quantum Field Theory, this equation (or rather a variant of it) also appears in *metastability* theory. For example, let us consider (1.1) for $\mathfrak{m} = 1$ and ξ replaced by $\sqrt{\varepsilon}\xi$, for $\varepsilon \in (0, 1)$. Then one retrieves the stochastic Allen-Cahn equation given by

$$(\partial_t - \Delta)X = -X^3 + X + \sqrt{2\varepsilon}\xi. \quad (1.4)$$

For small values of ε the solution is expected to spend long time stretches close to the minimisers of the potential (metastable states) of the deterministic system (i.e. $\varepsilon = 0$), with occasional noise-induced transitions between them. This phenomenon is known as metastability and it describes the behaviour of a system that spends long time in metastable states until it reaches equilibrium.

Early contributions in this framework were made in the 1-dimensional case. For example in [FJL82], Faris and Jona–Lasinio studied (1.4) on the level of large deviations. In [MS88, MOS89], Martinelli, Olivieri and Scoppola proved an asymptotic coupling for solutions that start close to the same minimiser of the potential of the deterministic system. A higher dimensional result appeared in [JLM90]. In this work, Jona–Lasinio and Mitter studied large deviations in $d = 2$ for the solution to (1.3), constructed in their previous work [JLM85] via Girsanov's transformation.

The main difficulty in (1.1) arises from the irregularity of space-time white noise ξ. In particular, the solution X is expected to be a Schwartz distribution when $d \geq 2$. Despite that we are allowed to multiply a distribution with a function under mild regularity assumptions, in general we cannot define products of arbitrary distributions; therefore it is unclear what the non-linear term X^3 represents.

A first, rather naive, attempt that might allow us to neglect the fact that the non-linear term X^3 is ill-defined, is to replace ξ by some smooth approximation ξ_N, $N \geq 1$, and study the limit as $N \to \infty$. In particular we can consider the system

$$(\partial_t - \Delta)X^N = -\left(X^N\right)^3 + \mathfrak{m}X^N + \sqrt{2}\xi_N. \tag{1.5}$$

Notice that for fixed N the non-linear term in the above equation is well-defined since the solution is actually a smooth function. We can then ask ourselves whether the approximation X^N converges to a non-trivial limit in the space of Schwartz distributions. However, it has already been established in [HRW12] that the set of limiting points of (1.5) in $d = 2$ is the singleton $\{X = 0\}$. Such a result indicates that approximations of this form fail to provide a meaningful solution. This is not unreasonable. After all, it turns out that the non-linear term $\left(X^N\right)^3$ diverges in the limit and it is exactly because of this divergence that we cannot solve (1.1) as it stands.

1.1.1 The Case $d = 2$

In dimension $d = 2$ (namely, in the case of the dynamic Φ^4_2) it is possible to cancel the divergence of the non-linear term $\left(X^N\right)^3$ in (1.5) if we instead consider the approximation

$$(\partial_t - \Delta)X^N = -\left(\left(X^N\right)^3 - 3\Re_N X^N\right) + \mathfrak{m}X^N + \sqrt{2}\xi_N. \tag{1.6}$$

Here $\Re_N$ is a divergent constant of the form $C_2 \log N + C_1$. The constant C_2 is explicit and depends on how we approximate ξ, while C_1 can be any real number, or even a sequence depending on N which converges in the limit.

The interpretation of solutions to (1.1) as limits of solutions to (1.6) in $d = 2$ was already apparent in [DPD03]. The extra term $-3\Re_N X^N$ in (1.6) can be interpreted as *Wick renormalisation*. Prior to the construction of the dynamics, Wick renormalisation had been used to rigorously construct the formal equilibrium (1.2). In this case a similar problem appears; the terms X^4 and X^2 are ill-defined. However, it had already been established by Nelson in [Nel73] that it can be circumvented using Wick renormalisation. The physical relevance of the approximation (1.6) was recently established in [MW17a] (see also [Ibe17] for a more general result), where it was proven that the limiting points of (1.5) arise naturally as suitably rescaled limits of the average magnetisation of an Ising–Kac model with Glauber dynamics close to critical temperature. The effect of the renormalisation constant $\mathfrak{R}_N$ corresponds to a shift of the critical temperature away from its mean field value. The 1-dimensional analogue of this result was shown in [BPRS93], while it was conjectured in [GLP99] for higher dimensions.

The first complete result on global existence and uniqueness appeared in [MW17c] by Mourrat and Weber, more than a decade after Da Prato and Debussche's breakthrough. In this work the authors went beyond initial conditions that are sampled by the invariant measure (which were previously considered in [DPD03]) and obtained global existence and uniqueness for arbitrary initial conditions of suitable regularity, first on arbitrary large tori and then on the whole plane via a periodisation technique. Their method was based on classical energy estimates which use the "good sign" of the cubic non-linearity.

Since the work of Mourrat and Weber [MW17c] appeared, equation (1.1) has been studied extensively in dimension $d = 2$ and in the last few years there is a tremendous amount of new results. For example, in the series of papers [RZZ17b, RZZ17a] the solutions from [MW17c] were identified with the solutions obtained by Dirichlet forms in [AR91]. The same work [RZZ17b, RZZ17a] also established ergodicity of solutions, that is, weak convergence of their law to a unique invariant measure. In [TW18b], a stronger result implied exponential mixing of the dynamics uniformly in the initial condition. Large deviations were studied in [HW15, CD17] and an Eyring–Kramers law for the transition times of Galerkin approximations to the solutions was established in [BDGW17]. Also, in [TW18a], the authors studied the long time behaviour of solutions for suitable initial conditions, path-wisely in the small noise regime.

1.1.2 Higher Dimensions

If we try to go just a step further and consider (1.1) in $d = 3$ the complexity of the problem increases significantly; to obtain a solution one needs to apply a more advanced renormalisation technique which cannot be interpreted as Wick renormalisation only. In particular, one needs to consider an approximation as in (1.6) where the constant $\mathfrak{R}_N$ is given by $C_3 N + C_2 \log N + C_1$. Here the part $C_3 N$ corresponds to Wick renormalisation in $d = 3$. However, the part $C_2 \log N$ cancels other divergences which appear because the space-time white noise ξ is more irregular in $d = 3$.

The first result about local existence and uniqueness in $d = 3$ appeared in the celebrated work by Hairer [Hai14] as an application of the theory of Regularity Structures. In [CC13] Catellier and Chouk retrieved the same result based on Paracontrolled Calculus developed in [GIP15]. Another approach was presented by Kupianen in [Kup16], who used the Wilsonian renormalisation group to obtain local existence and uniqueness. Later in [MW17b], Mourrat and Weber established global well-posedness based on the paracontrolled approach of [CC13, GIP15].

Since solution theories for (1.1) became available in $d = 3$, many other results have appeared. For example, discrete approximations were studied in [HM18a], large deviations in [HW15, CD17] (together with the case $d = 2$), continuity of the associated Markov semigroup in [HM18b] and universality in [FG17, SX18].

The important observation in $d = 3$ that allows us to prove well-posedness is the *subcriticality* of the equation. Roughly speaking, one expects that in small scales the solution to (1.1) behaves like the solution to the linear system, the additive stochastic heat equation. To understand this, let us formally apply the following rescaling,

$$(\bar{t}, \bar{z}) = (\lambda^2 t, \lambda z), \quad \bar{\xi} = \lambda^{\frac{d+2}{2}} \xi, \quad \bar{X} = \lambda^{\frac{d-2}{2}} X, \quad \bar{\mathfrak{m}} = \lambda^2 \mathfrak{m}.$$

Then (1.1) becomes

$$(\partial_t - \Delta)\bar{X} = -\lambda^{d-4} \bar{X}^3 + \bar{\mathfrak{m}} \bar{X} + \sqrt{2}\bar{\xi}$$

where $\bar{\xi}$ equals ξ in law. When $d < 4$ the non-linearity vanishes in the limit $\lambda \to 0$. When $d = 4$, (1.1) becomes critical and one does not expect to obtain any solution for $d \geq 4$. The theory of Regularity Structures as developed in the series of works [Hai14, BHZ16, CH16, BCCH17], provides local existence and uniqueness for a wide class of *subcritical* singular SPDEs. We refer the reader to [Hai14] for details about the importance of the subcriticality assumption in the solution theory for singular SPDEs.

1.2 The Stochastic Quantisation Equation

In the case $d = 2$ we can actually consider higher polynomial non-linearities. For example we can consider the equation

$$\big(\partial_t - (\Delta - 1)\big)X = -\sum_{k\le n} a_n X^k + \sqrt{2}\xi \tag{1.7}$$

where $n \ge 2$ is any odd number and $a_n > 0$. The restriction on the parameters n and a_n is only technical and allows to obtain global solutions in time. The dynamic Φ^4_2 is a special case of this equation for $n = 3$, $a_3 = 1$, $a_2, a_0 = 0$ and $a_1 = -(\mathfrak{m} + 1)$.

Equation (1.7) was introduced as the stochastic quantisation of the $\mathcal{P}(\varphi)_2$-Euclidean Quantum Field theory, which is described by the infinite dimensional measure

$$\nu(\,\mathrm{d}X) \propto \exp\left\{-\int\left(\frac{1}{2}|\nabla X(z)|^2 + \sum_{k\le n}\frac{\bar a_k}{k+1}X(z)^{k+1}\right)\,\mathrm{d}z\right\}\prod_z \mathrm{d}X(z). \tag{1.8}$$

Here $\bar a_k = a_k$, for every $k \ne 1$, and $\bar a_1 = a_1 + 1$ (the $+1$ comes from the term X on the left hand side of (1.7)). This measure is formally invariant for (1.7) and it was constructed by Nelson in [Nel73] using Wick renormalisation (see also [GRS75] and the references therein).

The right approximation to (1.7) (namely, the analogue of (1.6) in this case) is given by

$$\big(\partial_t - (\Delta - 1)\big)X^N = -\sum_{k\le n} a_k \mathcal{H}_k\left(X^N, \Re_N\right) + \sqrt{2}\xi_N. \tag{1.9}$$

Here $\Re_N$ is a suitable renormalisation constant which diverges logarithmically in N (same as in the case of the dynamic Φ^4_2 in Section 1.1.1) and $\mathcal{H}_n$ stands for the n-th Hermite polynomial given by the recursive formula

$$\begin{cases} \mathcal{H}_{-1}(X,C) = 0, \quad \mathcal{H}_0(X,C) = 1 \\ \mathcal{H}_n(X,C) = X\mathcal{H}_{n-1}(X,C) - (n-1)C\mathcal{H}_{n-2}(X,C) \end{cases}.$$

Let us mention that the extra terms appearing in (1.9) (but not in (1.7)) also correspond to Wick renormalisation, which in the case $d = 2$ works for arbitrary powers. However, in $d = 3$ one can only construct Wick powers up to order $n = 4$; we refer the reader to [CW17, Section 2.3.1] for a mathematical explanation on the failure of this construction in the case of powers of order $n \geq 5$. This is one of the reasons that (1.7) cannot be solved in $d = 3$ for $n > 3$ odd with any of the techniques developed in the last few years (for example [Hai14, GIP15]). Another reason is that the subcriticality assumption mentioned in Section 1.1.2 fails in this regime.

1.3 Main Results

The main focus of this thesis is to study properties of solutions to singular SPDEs. The model we consider here is the 2-dimensional stochastic quantisation equation (1.7) with periodic boundary conditions, since it is the simplest example of a family of singular SPDEs which can be solved using only Wick renormalisation techniques, that is, without any reference to more advanced solution theories. Another important feature of (1.7) is that solutions obtained via Wick renormalisation exist globally in time, and this allows to study their behaviour as $t \to \infty$.

The solutions of the equation are interpreted in a renormalised sense following the analysis in [DPD03] and [MW17c], which implies the following global well-posedness result for (1.7).

Theorem 1.1. *There exist a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and Banach (Besov) spaces* $(\mathcal{C}^{\beta}, \|\cdot\|_{\mathcal{C}^{\beta}}) \subset (\mathcal{C}^{-\alpha}, \|\cdot\|_{\mathcal{C}^{-\alpha}}) \subset (\mathcal{C}^{-\alpha_0}, \|\cdot\|_{\mathcal{C}^{-\alpha_0}})$ *such that the following conclusions hold.*

- *i. For every* $x \in \mathcal{C}^{-\alpha_0}$ *and suitable choice of renormalisation constant* $\mathfrak{R}_N$*, there exists a unique global-in-time limit* $X(\cdot; x)$ *of the solution* $X^N(\cdot; x)$ *to* (1.9) *with periodic boundary conditions and initial condition* x*. In particular, for every* $T > 0$ *the mapping* $\mathcal{C}^{-\alpha_0} \ni x \mapsto X(\cdot; x)$ *takes values in* $C((0, T]; \mathcal{C}^{-\alpha})$*,* $\mathbb{P}$*-almost surely.*

- *ii. If we let* $v(\cdot; x) = X(\cdot; x) - \upharpoonleft$*, where* $\upharpoonleft$ *solves the linear equation*

$$\big(\partial_t - (\Delta - 1)\big)\upharpoonleft = \sqrt{2}\xi,$$

  for every $T > 0$ *we have that* $\upharpoonleft \in C([0, T]; \mathcal{C}^{-\alpha})$ *and the mapping* $\mathcal{C}^{-\alpha_0} \ni x \mapsto v(\cdot; x)$ *takes values in* $C((0, T]; \mathcal{C}^{\beta})$*,* $\mathbb{P}$*-almost surely. Furthermore,* v *solves* *the remainder equation*

$$\big(\partial_t - (\Delta - 1)\big)v = -\sum_{k\le n} a_k \sum_{j\le k} \binom{k}{j} v^j \vee^{k-j}, \tag{1.10}$$

where $\vee^{k-j}$ *denotes the* $(k-j)$*-th Wick power of* $\uparrow$ *and for every* $T > 0$ *we have that* $\vee^{k-j} \in C([0,T];\mathcal{C}^{-\alpha})$.

iii. *The process* $X(\cdot;x)$ *is Markov and the associated semigroup is Feller.*

This theorem is essentially proved in Chapter 3.

Remark 1.2. To keep this introduction self-consistent we do not make the topologies $(\mathcal{C}^{-\alpha_0}, \|\cdot\|_{\mathcal{C}^{-\alpha_0}})$, $(\mathcal{C}^{-\alpha}, \|\cdot\|_{\mathcal{C}^{-\alpha}})$ and $(\mathcal{C}^{\beta}, \|\cdot\|_{\mathcal{C}^{\beta}})$ in Theorem 1.1 explicit. Let us only mention that $(\mathcal{C}^{-\alpha_0}, \|\cdot\|_{\mathcal{C}^{-\alpha_0}})$ and $(\mathcal{C}^{-\alpha}, \|\cdot\|_{\mathcal{C}^{-\alpha}})$ are Besov spaces of Schwartz distributions of negative regularity and $(\mathcal{C}^{\beta}, \|\cdot\|_{\mathcal{C}^{\beta}})$ is a Besov space of functions of positive regularity. We refer the reader to Theorems 2.1 and 3.12 for more details.

Remark 1.3. The Wick powers $\vee^{k-j}$ appearing in (1.10) are constructed in Section 2.1 using iterated stochastic integrals. In Section 2.2 we prove that they can be obtained as limits of suitably renormalised approximations.

In this thesis we study ergodicity and metastability of solutions to (1.7) as interpreted in Theorem 1.1. In the first case we study the long time behaviour of the law of the solutions. In the second, we study the long time behaviour of solutions for suitable initial conditions, path-wisely. The results to be presented here, have already appeared in the two original works [TW18b] and [TW18a] of the author.

1.3.1 Ergodicity

Following Parisi and Wu's original proposal [PW81], we prove that in the case of periodic boundary conditions the law of the solutions converges exponentially fast to a unique equilibrium uniformly in the initial condition. Our method consists of three different steps which are of independent interest.

The first step consists of proving a strong dissipative bound that holds uniformly in the initial condition. This bound reads as follows.

Theorem 1.4 (Theorem 4.1). *Let* $(\mathcal{C}^{-\alpha}, \|\cdot\|_{\mathcal{C}^{-\alpha}}) \subset (\mathcal{C}^{-\alpha_0}, \|\cdot\|_{\mathcal{C}^{-\alpha_0}})$ *and* $X(\cdot;x)$ *be as in Theorem 1.1. For every* $p \geq 2$

$$\sup_{x\in\mathcal{C}^{-\alpha_0}} \sup_{t\geq 0} (t\wedge 1)^{\frac{p}{n-1}} \mathbb{E}\|X(t;x)\|^p_{\mathcal{C}^{-\alpha}} < \infty,$$

where the parameter n *is the degree of the polynomial non-linearity in* (1.7).

The second step is the strong Feller property for the associated Markov semigroup. More precisely, we obtain the following theorem.

Theorem 1.5 (Theorem 4.13). *Let* $(\mathcal{C}^{-\alpha_0}, \|\cdot\|_{\mathcal{C}^{-\alpha_0}})$ *and* $X(\cdot;x)$ *be as in Theorem 1.1 and let* $P_t^*\delta_x$ *denote the law of* $X(t;x)$*. There exist* $\theta \in (0,1)$, $\sigma > 0$ *and* $C > 0$ *such that for every* $x \in \mathcal{C}^{-\alpha_0}$ *and* $y \in \{\bar{y} : \|\bar{y}-x\|_{\mathcal{C}^{-\alpha_0}} \leq 1\}$

$$\|P_t^*\delta_x - P_t^*\delta_y\|_{TV} \leq C(1+\|x\|)^\sigma \|x-y\|^\theta_{\mathcal{C}^{-\alpha_0}},$$

where $\|\cdot - \cdot\|_{TV}$ *denotes the total variation distance of probability measures on* $\mathcal{C}^{-\alpha_0}$.

The third step is the proof of a support theorem for the law of the solutions. This step is only implemented for $n = 3$ in (1.7) (see Remark 4.15) and reads as follows.

Theorem 1.6 (Corollary 4.18). *Let* $(\mathcal{C}^{-\alpha_0}, \|\cdot\|_{\mathcal{C}^{-\alpha_0}})$ *and* $X(\cdot;x)$ *be as in Theorem 1.1 for* $n = 3$ *in* (1.7). *For every* $x, y \in \mathcal{C}^{-\alpha_0}$ *and* $t, \varepsilon > 0$

$$\mathbb{P}(\|X(t;x) - y\|_{\mathcal{C}^{-\alpha_0}} < \varepsilon) > 0.$$

As a corollary of these results we obtain the following theorem which implies exponential mixing of the law of the solutions.

Theorem 1.7 (Corollary 4.20). *Let* $(\mathcal{C}^{-\alpha_0}, \|\cdot\|_{\mathcal{C}^{-\alpha_0}})$ *and* $X(\cdot;x)$ *be as in Theorem 1.1 for* $n = 3$ *in* (1.7) *and let* $P_t^*\delta_x$ *denote the law of* $X(t;x)$*. There exist* $t_0, \lambda_0 > 0$ *and a probability measure* μ_* *supported on* $\mathcal{C}^{-\alpha_0}$ *such that*

$$\sup_{x\in\mathcal{C}^{-\alpha_0}} \|P_t^*\delta_x - \mu_*\|_{TV} \leq \mathrm{e}^{-\lambda_0 t}.$$

Here $\|\cdot - \cdot\|_{TV}$ *denotes the total variation distance of probability measures on* $\mathcal{C}^{-\alpha_0}$.

1.3.2 Metastability

Motivated by [MS88, MOS89] we prove an asymptotic coupling in the small noise regime for initial conditions close to the minimisers of the deterministic potential. In this part we restrict ourselves to the stochastic Allen-Cahn equation (1.4) which is a special case of (1.7) if we let $n = 3$, $a_3 = 1$, $a_2, a_0 = 0$, and $a_1 = -2$ and replace ξ by $\sqrt{\varepsilon}\xi$ for $\varepsilon \in (0,1)$.

Remark 1.8. By choosing all the parameters in such a way that (1.7) coincides with (1.4), Theorem 1.1 still holds, but the stochastic objects $\overset{k-j}{\vee}$ in (1.10) should be multiplied by $\varepsilon^{\frac{k-j}{2}}$.

The main result can be expressed as follows.

Theorem 1.9 (Theorem 5.3). *Let* $(\mathcal{C}^\beta, \|\cdot\|_{\mathcal{C}^\beta}) \subset (\mathcal{C}^{-\alpha_0}, \|\cdot\|_{\mathcal{C}^{-\alpha_0}})$ *and* $X(\cdot;x)$ *be as in Theorem 1.1, where all the parameters are chosen such that* (1.7) *coincides with* (1.4) *and the size of the torus is sufficiently small. For every* $\kappa > 0$ *there exists* $a_0, \delta_0, C > 0$ *and* $\varepsilon_0 \in (0,1)$ *such that for every* $\varepsilon \leq \varepsilon_0$

$$\inf_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0} \mathbb{P}\left(\sup_{\|y-x\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0} \frac{\|X(t;y) - X(t;x)\|_{\mathcal{C}^\beta}}{\|y-x\|_{\mathcal{C}^{-\alpha_0}}} \leq C\mathrm{e}^{-(2-\kappa)t},\ \forall t \geq 1 \right)$$
$$\geq (1 - \mathrm{e}^{-a_0/\varepsilon}).$$

Remark 1.10. The interesting fact about this result is that although the solutions of the equation are distribution-valued, the difference $X(t;y) - X(t;x)$ is measured in $(\mathcal{C}^\beta, \|\cdot\|_{\mathcal{C}^\beta})$ which is a Besov space of functions of strictly positive regularity. At first glance this seems suspicious, but rewriting the difference as $\Big(X(t;y) - \varepsilon^{\frac{1}{2}}\upharpoonleft(t)\Big) - \Big(X(t;x) - \varepsilon^{\frac{1}{2}}\upharpoonleft(t)\Big)$ we immediately see that this is indeed the case since, by Theorem 1.1 (see also Remark 1.8), $X(t;x) - \varepsilon^{\frac{1}{2}}\upharpoonleft(t) \in \mathcal{C}^\beta$ for every $t > 0$ and $x \in \mathcal{C}^{-\alpha_0}$.

Building on the analysis of [BDGW17], as a corollary of this theorem we prove an Eyring-Kramers law for the transition times of the solutions between the minimisers of the potential of the deterministic system. This result reads as follows.

Theorem 1.11 (Theorem 5.31). *Let* $(\mathcal{C}^{-\alpha}, \|\cdot\|_{\mathcal{C}^{-\alpha}}) \subset (\mathcal{C}^{-\alpha_0}, \|\cdot\|_{\mathcal{C}^{-\alpha_0}})$ *and* $X(\cdot;x)$ *be as in Theorem 1.1, where all the parameters are chosen such that* (1.7) *coincides with* (1.4) *and the size of the torus is sufficiently small. For a closed set* $B \subset$

$\mathcal{C}^{-\alpha}$ let $\tau_B(X(\cdot;x))$ denote its first hitting time by $X(\cdot;x)$. For every suitable neighbourhoods A of -1 in $\mathcal{C}^{-\alpha_0}$ and B of 1 in $\mathcal{C}^{-\alpha}$ there exist $c_+, c_- > 0$ and $\varepsilon_0 > 0$ such that for every $\varepsilon \leq \varepsilon_0$

$$\begin{aligned}
&\sup_{x\in A} \mathbb{E}\tau_B(X(\cdot;x)) \\
&\quad \leq \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2}\frac{|\lambda_k|}{\nu_k}\exp\left\{\frac{\nu_k-\lambda_k}{\lambda_k+2}\right\}}\mathrm{e}^{(V(0)-V(-1))/\varepsilon}\left(1+c_+\sqrt{\varepsilon}\right), \\
&\inf_{x\in A} \mathbb{E}\tau_B(X(\cdot;x)) \\
&\quad \geq \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2}\frac{|\lambda_k|}{\nu_k}\exp\left\{\frac{\nu_k-\lambda_k}{\lambda_k+2}\right\}}\mathrm{e}^{(V(0)-V(-1))/\varepsilon}\left(1-c_-\varepsilon\right).
\end{aligned}$$

Here $\lambda_k \equiv \lambda_k(L)$ and $\nu_k \equiv \nu_k(L)$ are the eigenvalues of $-\Delta - 1$ and $-\Delta + 2$ endowed with periodic boundary conditions and V denotes the potential of the deterministic system (obtained by letting $\varepsilon = 0$ in (1.4)).

1.4 Outline

In Chapter 2 we rigorously construct the solution of the stochastic heat equation and its Wick powers and study their finite dimensional approximations. In Chapter 3 we prove global existence and uniqueness of solutions to (1.7) using Wick renormalisation and establish the Markov property. In Chapters 4 and 5 we prove our main results on ergodicity and metastability, respectively. Finally, some useful results that we repeatedly use in this thesis and some technical proofs can be found in the Appendix.

1.5 Notation

We let $\mathbb{T}^2 = \mathbb{R}^2/L\mathbb{Z}^2$ be the 2-dimensional torus of size L^2 for some $L > 0$. We denote by $\mathcal{C}^\infty$ the space of real-valued smooth functions over $\mathbb{T}^2$ and by $\mathcal{C}^\infty(\mathbb{R}^2)$ the space of real-valued smooth functions over $\mathbb{R}^2$. We denote by $\mathscr{S}'$ the dual space of Schwartz distributions acting on $\mathcal{C}^\infty$. For $p \in [1,\infty]$ we furthermore denote by L^p the space of p-integrable functions on $\mathbb{T}^2$, with the usual norm

$$\|f\|_{L^p} := \left(\int_{\mathbb{T}^2} |f(z)|^p \,\mathrm{d}z\right)^{\frac{1}{p}}$$

and the usual interpretation for $p = \infty$.

Although we only deal with spaces of real-valued functions, we prefer to work with the orthonormal basis $\{e_m\}_{m\in\mathbb{Z}^2}$ of trigonometric functions

$$e_m(z) := L^{-2} \mathrm{e}^{2\pi \mathrm{i} m\cdot z/L},$$

for $z \in \mathbb{T}^2$. Thus some complex-valued functions appear and we write

$$\langle f, g\rangle = \int_{\mathbb{T}^2} f(z)\overline{g(z)}\,\mathrm{d}z$$

for their inner product. With this notation, for $f \in L^2$, the m-th Fourier coefficient is given by

$$\hat{f}(m) := \langle f, e_m\rangle$$

and since f is real-valued we have the symmetry condition

$$\hat{f}(-m) = \overline{\hat{f}(m)}, \tag{1.11}$$

for any $m \in \mathbb{Z}^2$. For $f \in \mathscr{S}'$ we define the m-th Fourier coefficient as

$$\hat{f}(m) := \langle f, L^{-2}\cos(2\pi \mathrm{i} m \cdot /L)\rangle + \mathrm{i}\langle f, L^{-2}\sin(2\pi \mathrm{i} m\cdot /L)\rangle,$$

with the convention that $\langle f, \cdot\rangle$ stands for the action of f on $\mathcal{C}^\infty$.

For $\zeta \in \mathbb{R}^2$ and $r > 0$ we denote by $B(\zeta; r)$ the ball of radius r centred at ζ. We consider the annulus $\mathcal{A} = B\left(0; \frac{8}{3}\right) \setminus B\left(0; \frac{3}{4}\right)$ and a dyadic partition of unity $(\chi_\kappa)_{\kappa\geq -1}$ such that

i. $\chi_{-1} = \tilde{\chi}$ and $\chi_\kappa = \chi(\cdot/2^\kappa)$, $\kappa \geq 0$, for two radial functions $\tilde{\chi}, \chi \in \mathcal{C}^\infty(\mathbb{R}^2)$.

ii. $\operatorname{supp}\tilde{\chi} \subset B\left(0; \frac{4}{3}\right)$ and $\operatorname{supp}\chi \subset \mathcal{A}$.

iii. $\tilde{\chi}(\zeta) + \sum_{\kappa=0}^{\infty}\chi(\zeta/2^\kappa) = 1$, for every $\zeta \in \mathbb{R}^2 \setminus \{0\}$.

We furthermore let

$$\mathcal{A}_{2^\kappa} := 2^\kappa \mathcal{A}, \quad \kappa \geq 0.$$

Notice that $\operatorname{supp}\chi_\kappa \subset A_{2^\kappa}$, for every $\kappa \geq 0$. We also keep the convention that $\mathcal{A}_{2^{-1}} = B\left(0; \frac{4}{3}\right)$. The existence of such a dyadic partition of unity is given by [BCD11, Proposition 2.10].

For a function $f \in \mathcal{C}^\infty$ we define the κ-th Littlewood-Paley block as

$$\delta_\kappa f(z) := \sum_{m \in \mathbb{Z}^2} \chi_\kappa(m) \hat{f}(m) L^{-2} \mathrm{e}^{2\pi \mathrm{i} m \cdot z/L}, \quad \kappa \geq -1. \tag{1.12}$$

Sometimes it is convenient to write (1.12) as $\delta_\kappa f = \eta_\kappa * f$, $\kappa \geq -1$, where

$$\eta_\kappa * f(\cdot) = \int_{\mathbb{T}^2} \eta_\kappa(\cdot - z) f(z) \, \mathrm{d}z,$$

and

$$\eta_\kappa(z) := \sum_{m \in \mathbb{Z}^2} \chi_\kappa(m) L^{-2} \mathrm{e}^{2\pi \mathrm{i} m \cdot z/L}.$$

We are now ready to define the Besov space $\mathcal{B}^\alpha_{p,q}$ which we use to measure the regularity of Schwartz distributions.

Definition 1.12. For $\alpha \in \mathbb{R}$ and $p, q \in [1, \infty]$ we define the inhomogeneous periodic Besov norm as

$$\|f\|_{\mathcal{B}^\alpha_{p,q}} := \left\| \left(2^{\alpha\kappa} \|\delta_\kappa f\|_{L^p} \right)_{\kappa \geq -1} \right\|_{\ell^q}. \tag{1.13}$$

The Besov space $\mathcal{B}^\alpha_{p,q}$ is defined as the completion of $\mathcal{C}^\infty$ with respect to the norm (1.13). For simplicity we write $\mathcal{C}^\alpha$ to denote the Besov space $\mathcal{B}^\alpha_{\infty,\infty}$.

We would like to point out that for $p = q = \infty$ our definition of Besov spaces differs from the standard definition as the set of those distributions for which (1.13) is finite. Our convention has the advantage that all Besov spaces are separable. Many useful results about Besov spaces that we repeatedly use in this thesis can be found in Appendix A.

Throughout this thesis, C denotes a positive constant which changes from line to line but we make the dependence on different parameters explicit where necessary. Furthermore, through the proofs of our statements, in cases where we do not want to keep track of the various constants in the inequalities we use $\lesssim$ instead of $\leq C$. Finally, we use $a \vee b$ and $a \wedge b$ to denote the maximum and the minimum of a and b.

Chapter 2

The Stochastic Heat Equation and its Wick Powers

This is a preliminary chapter where we present the necessary stochastic tools to handle (1.7). In Section 2.1 we construct the solution of the stochastic heat equation and its Wick powers in terms of abstract iterated stochastic integrals in the spirit of [Nua06, Chapter 1]. In Section 2.2 we describe how these iterated stochastic integrals arise as limits of powers of solutions to finite dimensional approximations after renormalisation.

2.1 Construction of Wick Powers

Let ξ be a space-time white noise on $\mathbb{R} \times \mathbb{T}^2$ (see Appendix B for definitions) on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is fixed from now on. We set

$$\tilde{\mathcal{F}}_t = \sigma\left(\left\{\xi(\phi) : \phi|_{(t,+\infty)\times\mathbb{T}^2} \equiv 0,\, \phi \in L^2(\mathbb{R} \times \mathbb{T}^2)\right\}\right), \tag{2.1}$$

for $t > -\infty$ and denote by $(\mathcal{F}_t)_{t>-\infty}$ the usual augmentation (as in [RY99, Chapter 1.4]) of the filtration $(\tilde{\mathcal{F}}_t)_{t>-\infty}$.

For $s \in (-\infty, \infty)$ we consider the stochastic heat equation on $[s, \infty) \times \mathbb{T}^2$ with zero initial condition at time s given by

$$\begin{aligned} &\big(\partial_t - (\Delta - 1)\big)\stackrel{\bullet}{|}_s = \sqrt{2}\xi \\ &\stackrel{\bullet}{|}_s\big|_{t=s} = 0 \end{aligned} . \tag{2.2}$$

There are several ways to give a meaning to this equation. We simply use Duhamel's principle (see [Eva10, Section 2.3]) as a definition and for every

$\phi \in \mathcal{C}^\infty$ and $s < t$ we define

$$\langle \overset{\bullet}{\mid}_s(t), \phi \rangle := \sqrt{2} \int_s^t \int_{\mathbb{T}^2} \langle \phi, H(t-r, z-\cdot) \rangle \, \xi(\,\mathrm{d}r,\, \mathrm{d}z), \tag{2.3}$$

where $H(r, \cdot)$, $r \in \mathbb{R} \setminus \{0\}$, stands for the periodic heat kernel on L^2 given by

$$H(r, z) := \sum_{m \in \mathbb{Z}^2} \mathrm{e}^{-(1+4\pi^2|m|^2)r} e_m(z), \tag{2.4}$$

for all $z \in \mathbb{T}^2$. We furthermore let

$$S_1(t) := \mathrm{e}^{-t}\mathrm{e}^{t\Delta}$$

be the semigroup associated to the generator $\Delta - 1$ in L^2, i.e. the convolution operator with respect to the space variable $z \in \mathbb{T}^2$ with the kernel $H(t, \cdot)$.

The integral in (2.3) is a stochastic integral (see Appendix B for definitions) and for fixed $s < t$, $\overset{\bullet}{\mid}_s(t)$ is a family of Gaussian random variables indexed by $\mathcal{C}^\infty$.

It is more convenient to work with stationary processes, hence we extend definition (2.3) for $s = -\infty$. For $\phi \in \mathcal{C}^\infty$, $n \geq 2$ and $t > -\infty$ we also consider the iterated stochastic integral (see Appendix B for definitions) given by

$$\begin{aligned} &\langle \overset{n}{\vee}_{-\infty}(t), \phi \rangle \\ &\quad := 2^{\frac{n}{2}} \int_{\{(-\infty,t] \times \mathbb{T}^2\}^n} \left\langle \phi, \prod_{k=1}^n H(t-s_k, z_k - \cdot) \right\rangle \xi\left(\otimes_{k=1}^n \mathrm{d}s_k, \otimes_{k=1}^n \mathrm{d}z_k\right). \end{aligned} \tag{2.5}$$

We call $\overset{n}{\vee}_{-\infty}$ the n-th Wick power of $\overset{\bullet}{\mid}_{-\infty}$ and recall that for every $n \geq 1$ and $\phi \in \mathcal{C}^\infty$, $\langle \overset{n}{\vee}_{-\infty}(\cdot), \phi \rangle$ is an element in the n-th homogeneous Wiener chaos (see Appendix B for definitions). We furthermore point out that $\langle \overset{n}{\vee}_{-\infty}(\cdot), \phi \rangle$ is stationary, for every $\phi \in \mathcal{C}^\infty$.

The next theorem collects the optimal regularity properties of the processes $\{\overset{n}{\vee}_{-\infty}(\cdot)\}$, $n \geq 1$ and is very similar to the bounds originally derived in [DPD03, Lemma 3.2]. The precise statement is a consequence of the Kolmogorov-type criterion [MW17c, Lemma 9, Lemma 10] and the proof follows similar lines to the one of [MW17c, Theorem 5.1].

Theorem 2.1. *Let $p \geq 2$. For every $n \geq 1$ and $t_0 > -\infty$, the process $\overset{n}{\vee}_{-\infty}(t_0 + \cdot)$ admits a modification $\widetilde{\overset{n}{\vee}}_{-\infty}(t_0 + \cdot)$ such that $\widetilde{\overset{n}{\vee}}_{-\infty}(t_0 + \cdot) \in C\left([0, T]; \mathcal{C}^{-\alpha}\right)$, for* *every $T > 0$ and $\alpha > 0$. Furthermore, there exists $\theta \equiv \theta(\alpha) \in (0,1) > 0$ and $C \equiv C(T,\alpha,p)$ such that*

$$\mathbb{E} \sup_{s,t\in[0,T]} \frac{\|\widetilde{\overset{n}{\vee}}_{-\infty}(t_0+t) - \widetilde{\overset{n}{\vee}}_{-\infty}(t_0+s)\|^p_{\mathcal{C}^{-\alpha}}}{|t-s|^{p\theta}} \leq C. \tag{2.6}$$

For notational convenience we always refer to $\widetilde{\overset{n}{\vee}}_{-\infty}$ as $\overset{n}{\vee}_{-\infty}$.

Proof. See Appendix D. $\square$

Notice that for every $t \geq s$ we have that

$$\uparrow_s(t) = \uparrow_{-\infty}(t) - S_1(t-s)\uparrow_{-\infty}(s).$$

It is then reasonable to define (see also [MW17c, pp. 2442] for equivalent definitions) the n-th shifted Wick power of $\uparrow_s(t)$, $t > s > -\infty$, as

$$\overset{n}{\vee}_s(t) := \sum_{k=0}^{n} \binom{n}{k} (-1)^k \Big(S_1(t-s)\uparrow_{-\infty}(s)\Big)^k \overset{n-k}{\vee}_{-\infty}(t). \tag{2.7}$$

Here and below we use the convention $\overset{k}{\vee}_s(t) \equiv 1$ for $k = 0$ and for every $-\infty \leq s < t$. For simplicity, for every $n \geq 1$ and $t > 0$ we also write $\overset{n}{\vee}(t)$ instead of $\overset{n}{\vee}_0(t)$. We furthermore point out that the n-th shifted Wick power is not an element of the n-th homogeneous Wiener chaos (see Appendix B for definitions). We refer the reader to Proposition 2.3 below for a natural approximation of the objects defined in (2.7).

At this point we would like to mention that one might work directly with $\overset{n}{\vee}_{-\infty}$ instead of introducing (2.7) (see for example [DPD03] and [Hai14]). This alternative approach has the advantage that the diagrams are stationary in time. However, we prefer to work with (2.7) (as in [MW17c]) because when proving the Markov property (see Section 3.5) we use heavily that $\overset{n}{\vee}_s(t)$ is independent of $\mathcal{F}_s$ for any $s < t$ (see Proposition 2.3). A slight disadvantage of our convention is the logarithmic divergence of $\overset{n}{\vee}_s(t)$ as $t \downarrow s$ (see (2.8)).

The next proposition uses the regularisation property of the heat semigroup (see Proposition A.5) to show that for every $t > s$ and $n \geq 2$, $\overset{n}{\vee}_s(t)$ is a well-defined element in a Besov space of negative regularity close to 0.

Proposition 2.2. *Let $p \geq 2$ and $T > 0$. For every $s_0 > -\infty$, $\alpha \in (0,1)$ and $\alpha' > 0$ there exist $\theta \equiv \theta(\alpha, \alpha') > 0$ and $C \equiv C(T, \alpha, \alpha', p, n)$ such that*

$$\mathbb{E} \sup_{0 \leq s \leq t} \left(s^{(n-1)\alpha' p} \|\overset{n}{\vee}_{s_0}(s_0 + s)\|^p_{\mathcal{C}^{-\alpha}} \right) \leq C t^{p\theta}, \tag{2.8}$$

for every $t \leq T$.

Proof. We only show (2.8) for $s_0 = 0$. The proof of general $s_0 > -\infty$ follows similarly.

Let $\bar{\alpha} < \alpha \wedge \frac{2}{3}\alpha'$ and $V(s) = S_1(s)\left(-\uparrow_{-\infty}(0)\right)$. Using (A.1) and Propositions A.8 and A.5 we have that

$$\|V(s)^n\|_{\mathcal{C}^{-\alpha}} \lesssim \|V(s)\|^{n-1}_{\mathcal{C}^{2\bar{\alpha}}} \|V(s)\|_{\mathcal{C}^{-\bar{\alpha}}} \lesssim s^{-(n-1)\frac{3}{2}\bar{\alpha}} \|\uparrow_{-\infty}(0)\|^n_{\mathcal{C}^{-\bar{\alpha}}}.$$

In a similar way, for $k \notin \{0, n\}$, we have that

$$\|V(s)^k \overset{n-k}{\vee}_{-\infty}(s)\|_{\mathcal{C}^{-\alpha}} \lesssim s^{-k\frac{3}{2}\bar{\alpha}} \|\uparrow_{-\infty}(0)\|^k_{\mathcal{C}^{-\bar{\alpha}}} \|\overset{n-k}{\vee}_{-\infty}(s)\|_{\mathcal{C}^{-\bar{\alpha}}}.$$

Thus

$$\begin{aligned}
&\|\overset{n}{\vee}(s)\|_{\mathcal{C}^{-\alpha}} \\
&\quad \lesssim s^{-(n-1)\frac{3}{2}\bar{\alpha}} \|\uparrow_{-\infty}(0)\|^n_{\mathcal{C}^{-\bar{\alpha}}} + \sum_{k=0}^{n-1} \binom{n}{k} s^{-k\frac{3}{2}\bar{\alpha}} \|\uparrow_{-\infty}(0)\|^k_{\mathcal{C}^{-\bar{\alpha}}} \|\overset{n-k}{\vee}_{-\infty}(s)\|_{\mathcal{C}^{-\bar{\alpha}}}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbb{E} \sup_{0 \leq s \leq t} s^{(n-1)\alpha' p} \|\overset{n}{\vee}(s)\|^p_{\mathcal{C}^{-\alpha}} \\
&\quad \lesssim \sum_{k=0}^{n-1} \binom{n}{k} t^{\left((n-1)\alpha' - k\frac{3}{2}\bar{\alpha}\right)p} \left(\mathbb{E}\|\uparrow_{-\infty}(0)\|^{2kp}_{\mathcal{C}^{-\bar{\alpha}}} \right)^{\frac{1}{2}} \left(\mathbb{E} \sup_{0 \leq s \leq t} \|\overset{n-k}{\vee}_{-\infty}(s)\|^{2p}_{\mathcal{C}^{-\bar{\alpha}}} \right)^{\frac{1}{2}} \\
&\qquad + t^{(n-1)\left(\alpha' - \frac{3}{2}\bar{\alpha}\right)p} \mathbb{E}\|\uparrow_{-\infty}(0)\|^{np}_{\mathcal{C}^{-\bar{\alpha}}},
\end{aligned}$$

where we also use a Cauchy–Schwarz inequality to split the expectations in the sum. Combining with (2.6) we finally obtain (2.8). $\square$

2.2 Finite Dimensional Approximations

Let $\rho_N(z) = \sum_{|m|<N} e_m(z)$, $z \in \mathbb{T}^2$. For $t > s$ we define a finite dimensional approximation of $\uparrow_s(t)$ by

$$\uparrow^N_s(t, z) := \langle \uparrow_s(t), \rho_N(z - \cdot) \rangle.$$

We introduce the renormalisation constant

$$\Re_N := \mathbb{E}\upharpoonleft^N_{-\infty}(t,z)^2 = \|\mathbf{1}_{[0,\infty)}H_N\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2)}, \tag{2.9}$$

where $H_N(r,z) = (H(r,\cdot) * \rho_N)(z)$, noting that $\Re_N \approx \log N$ as $N \to \infty$. The expectation above is independent of t and z since $\upharpoonleft^N_{-\infty}(t,z)$ is stationary in t and z. For any integer $n \geq 1$, $s \geq -\infty$ and $z \in \mathbb{T}^2$ we also define

$$\overset{n}{\triangledown}{}^N_s(t,z) := \mathcal{H}_n\left(\upharpoonleft^N_s(t,z), \Re_N\right),$$

where $\mathcal{H}_n(X,C)$, $X, C \in \mathbb{R}$, stands for the n-th Hermite polynomial given by the recursive formula

$$\begin{cases} \mathcal{H}_{-1}(X,C) = 0, \quad \mathcal{H}_0(X,C) = 1 \\ \mathcal{H}_n(X,C) = X\mathcal{H}_{n-1}(X,C) - (n-1)C\mathcal{H}_{n-2}(X,C) \end{cases}. \tag{2.10}$$

The first three Hermite polynomials are given by $\mathcal{H}_1(X,C) = X$, $\mathcal{H}_2(X,C) = X^2 - C$, $\mathcal{H}_3(X,C) = X^3 - 3CX$.

We have the following approximation result.

Proposition 2.3. *Let $\alpha, \alpha' > 0$. Then for every $n \geq 1$ and $p \geq 2$ we have that*

$$\lim_{N\to\infty} \mathbb{E} \sup_{0\leq t\leq T} \|\overset{n}{\triangledown}_{-\infty}(s+t) - \overset{n}{\triangledown}{}^N_{-\infty}(s+t)\|^p_{\mathcal{C}^{-\alpha}} = 0,$$

$$\lim_{N\to\infty} \mathbb{E} \sup_{0\leq t\leq T} t^{(n-1)\alpha' p}\|\overset{n}{\triangledown}_s(s+t) - \overset{n}{\triangledown}{}^N_s(s+t)\|^p_{\mathcal{C}^{-\alpha}} = 0,$$

for every $s > -\infty$. In particular, $\overset{n}{\triangledown}_s(s+\cdot)$ is independent of $\mathcal{F}_s$ and for $s_1, s_2 \neq -\infty$, $\overset{n}{\triangledown}_{s_1}(s_1+\cdot) \overset{law}{=} \overset{n}{\triangledown}_{s_2}(s_2+\cdot)$.

Proof. See Appendix E. □

The following corollary is a consequence of the last proposition.

Corollary 2.4. *For every $n \geq 1$ and $t, h > 0$ the following identity holds $\mathbb{P}$-almost surely,*

$$\overset{n}{\triangledown}(t+h) = \sum_{k=0}^{n} \binom{n}{k} \Big(S_1(h)\upharpoonleft(t)\Big)^k \overset{n-k}{\triangledown}_t(t+h). \tag{2.11}$$

Proof. It suffices to check (2.11) for $\overset{n}{\triangledown}{}^N(t+h)$. The result then follows from Proposition 2.3. □

Chapter 3

Existence of Solutions

3.1 Introduction

In this chapter we consider the following renormalised SPDE on $[0,\infty)\times\mathbb{T}^2$,

$$\begin{aligned}\big(\partial_t-(\Delta-1)\big)X&=-\sum_{k=0}^{n}a_k:X^k:+\sqrt{2}\xi\\ X|_{t=0}&=x\end{aligned}\quad . \tag{3.1}$$

Here $:X^k:$ stands for the k-th Wick power of X, $x\in\mathcal{C}^{-\alpha_0}$, $n\geq 3$ is an odd integer and the a_k's are real numbers with $a_n>0$.

Remark 3.1. As in [DPD03] and [MW17c] we work directly with (3.1), which is the formal limit of the approximations (1.9). In later chapters (see Sections 4.3 and 5.3) we discuss finite dimensional approximations of the type (1.9), however for technical reasons the non-linearity is projected onto a suitable finite dimensional subspace.

Remark 3.2. Although we prefer to work with the massive Laplacian $\Delta-1$ in (3.1) one can consider any other mass by changing the value of the constant a_1 on the right hand side of (3.1).

Motivated by the Da Prato–Debussche method [DPD03] we give the following definition for solutions to (3.1).

Definition 3.3. We say that X solves (3.1) if $X=\upharpoonleft+v$, where $\upharpoonleft$ is the solution to (2.2) defined in (2.3) for $s=0$ and the remainder v is a mild solution of the following random PDE,

$$
\begin{aligned}
\big(\partial_t - (\Delta - 1)\big)v &= -\sum_{k=0}^{n} a_k \sum_{j=0}^{k} \binom{k}{j} v^j \, \bigvee^{k-j} . \\
v|_{t=0} &= x
\end{aligned}
\tag{3.2}
$$

Here $\bigvee^{k-j}$ stands for the $(k-j)$-th Wick power of $\uparrow$ defined in (2.7) for $s = 0$.

Remark 3.4. In [MW17c] $\uparrow$ is started from x and consequently there (3.2) is solved with zero initial condition. Our approach of starting $\uparrow$ from 0 and the remainder v from x has the advantage that the strong non-linear damping in (3.2) acts directly on the initial condition, yielding a strong a priori estimate for v that is independent of x (see Proposition 3.10).

Our aim is to prove global existence and uniqueness for the remainder equation (3.2) which is achieved by classical arguments from PDE theory. In the first step we prove local existence of solutions to (3.2) on suitable Besov spaces (see Theorem 3.6). Then, building on the analysis in [MW17c] and using a simple comparison test for non-linear ordinary differential equations we obtain an a priori estimate for v independent of the initial condition (see Proposition 3.10). This estimate is finally used to prove global existence (see Theorem 3.12). In these steps our analysis is deterministic and uses only the analytical properties of the Wick powers implied by Proposition 2.2. Finally, combining the results of Section 2.2 with a simple factorisation method we prove the Markov property for the process $X = \uparrow + v$ (see Proposition 3.14).

Let us mention that the Markov property for X was previously established in [RZZ17b] based on the identification of the dynamics with the solutions constructed via Dirichlet forms in [AR91]. However, our proof is based on standard arguments from SPDE Theory only, appearing in the classical book [DPZ92].

3.1.1 Outline

In Section 3.2 we prove local existence of solutions to (3.2). In Section 3.3 we prove an a priori estimate for solutions to (3.2). In Section 3.4 we prove global existence of solutions. Finally, in Section 3.5 we prove the Markov property for the solution of (3.1) as defined in Definition 3.3.

3.1.2 Notation

For convenience we rewrite (3.2) as

$$\begin{aligned} \big(\partial_t - (\Delta - 1)\big)v &= -\sum_{j=0}^{n} v^j Z^{(n-j)} \\ v|_{t=0} &= x \end{aligned}, \tag{3.3}$$

where

$$Z^{(n-j)} = \sum_{k=j}^{n} a_k \binom{k}{j} \overset{k-j}{\vee}, \tag{3.4}$$

for all $0 \le j \le n-1$ and $Z^{(0)} = a_n$. In this chapter $\underline{Z}$ always denotes a vector of the form $\big(Z^{(j)}\big)_{j=0}^{n}$.

We fix $\alpha_0 \in (0, \frac{1}{n})$ (to measure the regularity of the initial condition x in $\mathcal{C}^{-\alpha_0}$), $\beta > 0$ (to measure the regularity of the remainder v in $\mathcal{C}^{\beta}$) and $\gamma > 0$ (to measure the rate of blow-up of the $\|v(t)\|_{\mathcal{C}^\beta}$ for t close to 0, see e.g. Theorem 3.6) such that

$$\gamma < \frac{1}{n}, \quad \frac{\beta + \alpha_0}{2} < \gamma. \tag{3.5}$$

For an arbitrary $\alpha \in (0,1)$ we let

$$C^{n,-\alpha}(0;T) := C\left([0,T]; \mathcal{C}^{-\alpha}\right) \times C\left((0,T]; \mathcal{C}^{-\alpha}\right)^{n-1}. \tag{3.6}$$

For $\alpha, \alpha' > 0$ we also define a norm on $C^{n,-\alpha}(0;T)$ given by

$$|||\underline{Z}|||_{\alpha;\alpha';T} := \max_{k=1,2,\dots,n} \left\{ \sup_{0 \le t \le T} t^{(k-1)\alpha'} \|Z^{(k)}(t)\|_{\mathcal{C}^{-\alpha}} \right\}. \tag{3.7}$$

Notice that for every $\alpha \in (0,1)$, $\underline{Z} \in C^{n,-\alpha}(0;T)$, for every $T > 0$, and by (2.8) for every $\alpha' > 0$ there exists $\theta > 0$ such that

$$\mathbb{E}|||\underline{Z}|||^p_{\alpha;\alpha';t} \le C t^{p\theta} \tag{3.8}$$

for every $t \le T$, $p \ge 2$.

We also fix $\alpha < \alpha_0$ small enough (the precise value is fixed below in the proof of Theorem 3.6) and a norm $|||\cdot|||_{\alpha;\alpha';T}$ on $C^{n,-\alpha}(0;T)$, for some $\alpha' > 0$ but still sufficiently small. We furthermore let

$$F(v, \underline{Z}) := \sum_{j=0}^{n} v^j Z^{(n-j)}. \tag{3.9}$$

3.2 Local Existence

In this section we prove local in time existence of solutions to (3.3) and stability with respect to its various parameters.

Definition 3.5. Let $T > 0$ and $x \in \mathcal{C}^{-\alpha_0}$. We say that a function v is a mild solution to (3.3) up to time T if $v \in C\left((0,T];\mathcal{C}^\beta\right)$ and

$$v(t) = S_1(t)x - \int_0^t S_1(t-s)F(v(s),\underline{Z}(s))\,\mathrm{d}s,$$

for every $t \le T$.

We have the following local in time existence theorem.

Theorem 3.6 ([DPD03, Proposition 4.4], [MW17c, Theorem 6.2])**.** *Let $x \in \mathcal{C}^{-\alpha_0}$ and $R > 0$ such that $\|x\|_{\mathcal{C}^{-\alpha_0}} \le R$. Then for every $\beta, \gamma > 0$ satisfying* (3.5) *and $T > 0$ there exists $T_* \equiv T_*(R, |||\underline{Z}|||_{\alpha;\alpha';T}) \le T$ such that* (3.3) *has a unique mild solution on $[0,T_*]$ and*

$$\sup_{0\le s\le T_*} s^\gamma \|v(s)\|_{\mathcal{C}^\beta} \le 1.$$

If we furthermore assume that $|||\underline{Z}|||_{\alpha;\alpha';T} \le 1$, then there exists $\theta > 0$ and a constant $C > 0$ independent of R such that

$$T_* = \left(\frac{1}{C(R+1)}\right)^{\frac{1}{\theta}}. \tag{3.10}$$

Proof. This theorem is (essentially) proved in [MW17c, Theorem 6.2], but the expression (3.10) is not made explicit there; we give a sketch. It is sufficient to prove that for T_* as in (3.10) the operator

$$\mathscr{M}_{T_*}v(t) = S_1(t)x + \int_0^t S_1(t-s)F(v(s),\underline{Z}(s))\,\mathrm{d}s$$

is a contraction on the set $\mathscr{B}_{T_*} := \{\sup_{0\le s\le T_*} s^\gamma\|v(s)\|_{\mathcal{C}^\beta} \le 1\}$, that is, we need to show that $\mathscr{M}_{T_*}$ maps $\mathscr{B}_{T_*}$ into itself and that for $v, \tilde{v} \in \mathscr{B}_{T_*}$ we have

$$\sup_{0\le s\le T_*} s^\gamma\|\mathscr{M}_{T_*}v(s) - \mathscr{M}_{T_*}\tilde{v}(s)\|_{\mathcal{C}^\beta} \le (1-\lambda) \sup_{0\le s\le T_*} s^\gamma\|v(s)-\tilde{v}(s)\|_{\mathcal{C}^\beta}$$

for some $\lambda > 0$. We only show the first property. We first notice that using the explicit form of F (see (3.9))

$$\begin{aligned}\|\mathscr{M}_{T_*}v(s)\|_{\mathcal{C}^\beta} &\lesssim \|S_1(t)x\|_{\mathcal{C}^\beta} + \int_0^t \|S_1(t-s)v(s)^n\|_{\mathcal{C}^\beta}\,\mathrm{d}s \\ &\quad + \sum_{j=0}^{n-1}\int_0^t \|S_1(t-s)v(s)^j Z^{(n-j)}(s)\|_{\mathcal{C}^\beta}\,\mathrm{d}s \\ &\lesssim t^{-\frac{\beta+\alpha_0}{2}}\|x\|_{\mathcal{C}^{-\alpha_0}} + \int_0^t s^{-n\gamma}\,\mathrm{d}s + \int_0^t (t-s)^{-\frac{\alpha+\beta}{2}} s^{-(n-1)\gamma}\,\mathrm{d}s,\end{aligned}$$

where we use Proposition A.5 and we furthermore assumed that $\alpha' < \gamma$. By (3.5) if we choose $\alpha > 0$ sufficiently small so that $\frac{\alpha+\beta}{2} + (n-1)\gamma < 1$ we get

$$\|\mathscr{M}_{T_*}v(t)\|_{\mathcal{C}^\beta} \lesssim t^{-\frac{\beta+\alpha_0}{2}}\|x\|_{\mathcal{C}^{-\alpha_0}} + t^{1-n\gamma} + t^{1-\frac{\alpha+\beta}{2}-(n-1)\gamma}$$

and multiplying both sides by t^γ we obtain that

$$t^\gamma\|\mathscr{M}_{T_*}v(t)\|_{\mathcal{C}^\beta} \lesssim t^{\gamma-\frac{\beta+\alpha_0}{2}}R + t^{1-(n-1)\gamma} + t^{1-\frac{\alpha+\beta}{2}-(n-2)\gamma} \lesssim t^\theta(R+1).$$

Then, for $T_* \equiv T_*(R)$ as in (3.10) and every $t \le T_*$ we get that

$$\sup_{0\le s\le t} s^\gamma \|\mathscr{M}_{T_*}v(s)\|_{\mathcal{C}^\beta} \le 1,$$

which implies that $\mathscr{M}_{T_*}$ maps $\mathscr{B}_{T_*}$ into itself. □

The next proposition is a stability result which we use later on in Section 4.3. We first introduce some extra notation. Let $\{\underline{Z}^N\}_{N\ge1}$ take values in $C^{n,-\alpha}(0;T)$ such that

$$\lim_{N\to\infty} |||\underline{Z}^N - \underline{Z}|||_{\alpha;\alpha';T} = 0.$$

Furthermore, let $F_N = \hat{\Pi}_N F$, where $\hat{\Pi}_N$ is a linear smooth approximation such that the following properties hold for every $\lambda \in \mathbb{R}$,

i. $\|\hat{\Pi}_N\|_{\mathcal{C}^\lambda\to\mathcal{C}^\lambda} \le C$, for every $N \ge 1$.

ii. For every $\delta > 0$ there exists $\theta \equiv \theta(\lambda,\delta)$ such that

$$\|\hat{\Pi}_N x - x\|_{\mathcal{C}^{\lambda-\delta}} \le C\frac{1}{N^\theta}\|x\|_{\mathcal{C}^\lambda}.$$

One can check that $\hat{\Pi}_N = \sum_{-1\le\kappa<\log_2 N} \delta_\kappa$ is such a linear smooth approximation.

Denote by v^N the corresponding mild solution of (3.3) with F replaced by F_N, $\underline{Z}$ by $\underline{Z}^N$ and initial condition $x_N = \hat{\Pi}_N x$ (short time existence of v^N is ensured by the same arguments as in the proof of [MW17c, Theorem 6.2]). We then have the following proposition.

Proposition 3.7. *Let v be the unique solution of* (3.3) *on a closed interval $[0, T_*]$ (i.e. the solution does not explode at T_*). Then for every $N \geq 1$ there exists a unique solution v^N to the approximate equation up to some (possibly infinite) explosion time T_*^N. Furthermore, there exists $N_0 \geq 1$ such that for every $N \geq N_0$, $T_*^N \geq T_*$, and we have that*

$$\lim_{N\to\infty} \sup_{0\le t\le T_*^N\wedge T_*} t^\gamma \|v(t) - v^N(t)\|_{\mathcal{C}^\beta} = 0.$$

Proof. By (3.5) it is possible to find $\delta > 0$ such that

$$\frac{\delta}{2} + n\gamma < 1, \qquad \frac{\alpha_0 + \delta + \beta}{2} + (n-1)\gamma < 1.$$

For $N \geq 1$ we notice that

$$\begin{aligned} &v(t) - v^N(t) \\ &\quad = S_1(t)\,(x - x_N) - \int_0^t S_1(t-s)\left(F(v(s), \underline{Z}(s)) - F_N(v^N(s), \underline{Z}^N(s))\right) \mathrm{d}s \end{aligned}$$

and using (A.7) and property ii of $\hat{\Pi}_N$ we get

$$\begin{aligned} &\|v(t) - v^N(t)\|_{\mathcal{C}^\beta} \\ &\quad \lesssim t^{-\frac{\alpha_0+\delta+\beta}{2}} \frac{1}{N^\theta} \|x\|_{\mathcal{C}^{-\alpha_0}} + \int_0^t (t-s)^{\frac{\delta}{2}} \|v(s)^n - \hat{\Pi}_N v^N(s)^n\|_{\mathcal{C}^{\beta-\delta}} \,\mathrm{d}s \\ &\qquad + \int_0^t (t-s)^{-\frac{\alpha+\delta+\beta}{2}} \|R(v(s), \underline{Z}(s)) - R_N(v^N(s), \underline{Z}^N(s))\|_{\mathcal{C}^{-\alpha-\delta}} \,\mathrm{d}s, \end{aligned}$$

where $R(v, \underline{Z}) = \sum_{j=0}^{n-1} v^j Z^{(n-j)}$ and $R_N = \hat{\Pi}_N R$. Using the triangle inequality as well as the properties i and ii of $\hat{\Pi}_N$ we have that

$$\|v(s)^n - \hat{\Pi}_N v^N(s)^n\|_{\mathcal{C}^{\beta-\delta}} \lesssim \frac{1}{N^\theta} \|v^N(s)^n\|_{\mathcal{C}^\beta} + \|v(s)^n - v^N(s)^n\|_{\mathcal{C}^\beta}$$

and

$$
\begin{aligned}
&\|R(v(s),\underline{Z}(s)) - R_N(v^N(s),\underline{Z}^N(s))\|_{\mathcal{C}^{-\alpha-\delta}} \\
&\quad\lesssim \frac{1}{N^\theta}\|R(v(s),\underline{Z}(s))\|_{\mathcal{C}^{-\alpha}} + \|R(v(s),\underline{Z}(s)) - R(v^N(s),\underline{Z}(s))\|_{\mathcal{C}^{-\alpha}} \\
&\qquad + \|R(v^N(s),\underline{Z}(s)) - R(v^N(s),\underline{Z}^N(s))\|_{\mathcal{C}^{-\alpha}}.
\end{aligned}
$$

Let $M = \sup_{t\le T_*} t^\gamma \|v(s)\|_{\mathcal{C}^\beta}$, $L = \|\!|\underline{Z}|\!\|_{\alpha;\alpha';T}$ and $\iota_N = \inf\{t > 0, t \le T_*^N : t^\gamma\|v(t) - v^N(t)\|_{\mathcal{C}^\beta} > 1\}$. Then, for every $s \le \iota_N \wedge T_*$, we have the bounds

$$
\begin{aligned}
&\|v^N(s)^n\|_{\mathcal{C}^\beta} \le Cs^{-n\gamma}, \\
&\|v(s)^n - v^N(s)^n\|_{\mathcal{C}^\beta} \le Cs^{-n\gamma} \sup_{t\le\iota_N\wedge T_*} t^\gamma\|v(t) - v^N(t)\|_{\mathcal{C}^\beta},
\end{aligned}
$$

as well as

$$
\begin{aligned}
&\|R(v(s),\underline{Z}(s))\|_{\mathcal{C}^{-\alpha}} \le Cs^{-(n-1)\gamma} \\
&\|R(v(s),\underline{Z}(s)) - R(v^N(s),\underline{Z}(s))\|_{\mathcal{C}^{-\alpha}} \le Cs^{-(n-1)\gamma} \sup_{t\le\iota_N\wedge T_*} t^\gamma\|v(t) - v^N(t)\|_{\mathcal{C}^\beta} \\
&\|R(v^N(s),\underline{Z}(s)) - R(v^N(s),\underline{Z}^N(s))\|_{\mathcal{C}^{-\alpha}} \le Cs^{-(n-1)\gamma}\left|\!\left|\!\left|\underline{Z} - \underline{Z}^N\right|\!\right|\!\right|_{\alpha;\alpha';T},
\end{aligned}
$$

where the constant C depends on M and L. Thus

$$
\begin{aligned}
\|v(t) - v^N(t)\|_{\mathcal{C}^\beta} \le C\Bigg(&\frac{1}{N^\theta}\left(t^{-\frac{\alpha_0+\delta+\beta}{2}}\|x\|_{\mathcal{C}^{-\alpha_0}} + t^{1-\frac{\delta}{2}-n\gamma} + t^{1-\frac{\alpha+\delta+\beta}{2}-(n-1)\gamma}\right) \\
&+ \sup_{s\le\iota_N\wedge T_*} s^\gamma\|v(s) - v^N(s)\|_{\mathcal{C}^\beta}\left(t^{1-\frac{\delta}{2}-n\gamma}t^{1-\frac{\alpha+\delta+\beta}{2}-(n-1)\gamma}\right) \\
&+ \left|\!\left|\!\left|\underline{Z} - \underline{Z}^N\right|\!\right|\!\right|_{\alpha;\alpha';T} t^{1-\frac{\alpha+\delta+\beta}{2}-(n-1)\gamma}\Bigg).
\end{aligned}
$$

Multiplying by t^γ and choosing $\tilde{T}_* \equiv \tilde{T}_*(M,L) > 0$ sufficiently small we can assure that

$$
\sup_{t\le\tilde{T}_*} t^\gamma\|v(t) - v^N(t)\|_{\mathcal{C}^\beta} \le \frac{1}{N^\theta}\|x\|_{\mathcal{C}^{-\alpha_0}} + \left|\!\left|\!\left|\underline{Z} - \underline{Z}^N\right|\!\right|\!\right|_{\alpha;\alpha';T} + \frac{1}{N^\theta}.
$$

Iterating the procedure if necessary we find $M_* > 0$, independent of N since $\iota_N \wedge T_* \le T_*$, and $C > 0$ such that

$$
\begin{aligned}
&\sup_{t\le\iota_N\wedge T_*} t^\gamma\|v(t) - v^N(t)\|_{\mathcal{C}^\beta} \\
&\quad\le (M_*C + 1)\left(\frac{1}{N^\theta}\|x\|_{\mathcal{C}^{-\alpha_0}} + \left|\!\left|\!\left|\underline{Z} - \underline{Z}^N\right|\!\right|\!\right|_{\alpha;\alpha';T} + \frac{1}{N^\theta}\right). \qquad (3.11)
\end{aligned}
$$

Let $N_0 \geq 1$ such that for every $N \geq N_0$

$$\frac{1}{N^\theta}\|x\|_{\mathcal{C}^{-\alpha_0}} + \left|\left|\left|\underline{Z} - \underline{Z}^N\right|\right|\right|_{\alpha;\alpha';T} + \frac{1}{N^\theta} < \frac{1}{(M_* C + 1)}.$$

Then for every $N \geq N_0$

$$\sup_{t \leq \iota_N \wedge T_*} t^\gamma \|v(t) - v^N(t)\|_{\mathcal{C}^\beta} < 1$$

and the definition of ι_N implies that $\iota_N \wedge T_* = T_*$, which proves the first claim. For the second claim we just let $N \to \infty$ in (3.11). □

3.3 A Priori Estimates

We first need the following proposition from [MW17c] which is obtained by testing (3.3) with arbitrary odd powers of v.

Proposition 3.8 ([MW17c, Proposition 14]). *Let $v \in C\left((0,T];\mathcal{C}^\beta\right)$ be a mild solution to* (3.3). *For every $s_0 > 0$ and every even integer $p \geq 2$*

$$\begin{aligned} &\frac{1}{p}\left(\|v(t)\|_{L^p}^p - \|v(s_0)\|_{L^p}^p\right) \\ &\quad = \int_{s_0}^t \Big(-(p-1)\langle \nabla v(s), v(s)^{p-2}\nabla v(s)\rangle - \langle v(s), v(s)^{p-1}\rangle \\ &\qquad - \langle F(v(s), \underline{Z}(s)), v(s)^{p-1}\rangle \Big)\, \mathrm{d}s, \end{aligned} \tag{3.12}$$

for every $t \in [s_0, T]$. In particular, if we differentiate with respect to t,

$$\begin{aligned} \frac{1}{p}\partial_t \|v(t)\|_{L^p}^p &= -(p-1)\langle \nabla v(t), v(t)^{p-2}\nabla v(t)\rangle - \langle v(t), v(t)^{p-1}\rangle \\ &\qquad - \langle F(v(t), \underline{Z}(t)), v(t)^{p-1}\rangle, \end{aligned} \tag{3.13}$$

for every $t \in (0,T)$.

Remark 3.9. This proposition involves spatial derivatives of v up to first order and the proof of (3.12) requires some time regularity on v. Our local existence theory implies that $v \in C((0,T];\mathcal{C}^\beta)$ for some $\beta < 1$ (see (3.5)) due to the fact that we start (3.3) with initial condition in $\mathcal{C}^{-\alpha_0}$. This is the reason we state (3.12) for $s_0 > 0$. However one can prove that for fixed $t > 0$ v is almost a $\mathcal{C}^2$ function (see [MW17c, Theorem 6.2]), as well as a Hölder continuous function from $(0,T]$ to L^∞ (see [MW17c, Proposition 12]) for some exponent strictly greater that $\frac{1}{2}$.

Global existence of (3.3) for $x \in \mathcal{C}^\beta$ was already established in [MW17c] based on a priori estimates of the L^p norm of v. Here we derive a stronger bound which does not depend on the initial condition x.

Proposition 3.10. *Let $v \in C((0,T];\mathcal{C}^\beta)$ be a solution of (3.3) with initial condition $x \in \mathcal{C}^{-\alpha_0}$ and $p \geq 2$ be an even integer. Then for every $0 < t \leq T$ and $\lambda = \frac{p+n-1}{p}$*

$$\|v(t)\|_{L^p}^p \leq C \left[t^{-\frac{1}{\lambda-1}} \vee \left(\sum_{j,i} t^{-\alpha' p_i^j} \sup_{0\leq r\leq t} \left(r^{\alpha' p_i^j} \|Z^{(n-j)}(r)\|_{\mathcal{C}^{-\alpha}}^{p_i^j} \right) \right)^{\frac{1}{\lambda}} \right], \tag{3.14}$$

for some $p_i^j > 0$. In particular, the bound is independent from $\|x\|_{\mathcal{C}^{-\alpha_0}}$ and the randomness outside of the interval $[0,t]$.

Proof. Let

$$\alpha < \frac{1}{(p+n-1)(n-1)} \tag{3.15}$$

and recall that $F(v(s), \underline{Z}(s) = \sum_{j=0}^n v_s^j Z^{(n-j)}(s)$. Thus

$$\begin{aligned} \langle F(v(s), \underline{Z}(s)), v_s^{p-1} \rangle &= \sum_{j=0}^n \langle v(s)^{p+j-1}, Z^{(n-j)}(s) \rangle \\ &= a_n \|v(s)^{p+n-1}\|_{L^1} + \langle g(s), v(s)^{p-1} \rangle, \end{aligned}$$

where $g(s) = \sum_{j=0}^{n-1} v(s)^j Z^{(n-j)}(s)$, and we rewrite (3.13) as

$$\begin{aligned} &\frac{1}{p} \partial_s \|v(s)\|_{L^p}^p \\ &\quad = - \big((p-1)\|v(s)^{p-2}|\nabla v(s)|^2\|_{L^1} + a_n \|v(s)^{p+n-1}\|_{L^1} + \|v(s)^p\|_{L^1} \big) \\ &\qquad - \langle g(s), v(s)^{p-1} \rangle, \end{aligned} \tag{3.16}$$

for all $0 < s \leq t$, where we use that p is an even integer. Let

$$K(s) := \|v(s)^{p-2}|\nabla v(s)|^2\|_{L^1}, \quad L(s) := a_n \|v(s)^{p+n-1}\|_{L^1}. \tag{3.17}$$

The idea is to control the terms of $\langle g(s), v(s)^{p-1} \rangle$ by $K(s)$ and $L(s)$.

We start with the leading term of $\langle g(s), v(s)^{p-1} \rangle$, $\langle v(s)^{p+n-2}, Z^{(1)}(s) \rangle$. By Proposition A.9

$$|\langle v(s)^{p+n-2}, Z^{(1)}(s) \rangle| \lesssim \|v(s)^{p+n-2}\|_{\mathcal{B}_{1,1}^\alpha} \|Z^{(1)}(s)\|_{\mathcal{C}^{-\alpha}}. \tag{3.18}$$

Using (A.13)

$$\|v(s)^{p+n-2}\|_{\mathcal{B}^{\alpha}_{1,1}} \lesssim \|v(s)^{p+n-2}\|_{L^1}^{1-\alpha}\|v(s)^{p+n-3}|\nabla v(s)|\|_{L^1}^{\alpha} + \|v(s)^{p+n-2}\|_{L^1}. \tag{3.19}$$

We treat each term in (3.19) separately. We first notice that by Jensen's inequality $\|v(s)^{p+n-2}\|_{L^1} \lesssim L(s)^{\frac{p+n-2}{p+n-1}}$. For the gradient term, using the Cauchy-Schwarz inequality we obtain

$$\|v(s)^{p+n-3}|\nabla v(s)|\|_{L^1} \lesssim \|v(s)^{p+2(n-2)}\|_{L^1}^{\frac{1}{2}} K(s)^{\frac{1}{2}}. \tag{3.20}$$

Recall the Sobolev inequality

$$\|f\|_{L^q} \lesssim \left(\|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2\right)^{\frac{1}{2}},$$

for every $q < \infty$ (see [DNPV12, Section 6], [Eva10, Section 5.6] for Sobolev inequalities in the same spirit). In particular, for $q = \frac{2(p+2(n-2))}{p}$, we have that

$$\|v(s)^{\frac{p}{2}}\|_{L^q}^{\frac{q}{2}} \lesssim \|v(s)^{\frac{p}{2}}\|_{L^2}^{\frac{q}{2}} + \|\nabla v(s)^{\frac{p}{2}}\|_{L^2}^{\frac{q}{2}},$$

which implies

$$\|v(s)^{p+2(n-2)}\|_{L^1}^{\frac{1}{2}} \lesssim \|v(s)^p\|_{L^1}^{\frac{1}{2}+\frac{n-2}{p}} + K(s)^{\frac{1}{2}+\frac{n-2}{p}}, \tag{3.21}$$

where $\|v(s)^p\|_{L^1}^{\frac{1}{2}+\frac{n-2}{p}} \lesssim L(s)^{\frac{\frac{p}{2}+n-2}{p+n-1}}$ by Jensen's inequality. Combining (3.19), (3.20) and (3.21)

$$\begin{aligned}\|v(s)^{p+n-2}\|_{\mathcal{B}^{\alpha}_{1,1}} \lesssim K(s)^{\frac{\alpha}{2}} L(s)^{\frac{(p+n-2)-\frac{p}{2}\alpha}{p+n-1}} &+ K(s)^{\left(1+\frac{n-2}{p}\right)\alpha} L(s)^{\frac{(p+n-2)(1-\alpha)}{p+n-1}} \\ &+ L(s)^{\frac{p+n-2}{p+n-1}}. \end{aligned} \tag{3.22}$$

By (3.15) we notice that

$$\frac{\alpha}{2} + \frac{(p+n-2)-\frac{p}{2}\alpha}{p+n-1} < 1$$

and

$$\left(1+\frac{n-2}{p}\right)\alpha + \frac{(p+n-2)(1-\alpha)}{p+n-1} < 1,$$

thus we can find $\gamma_1, \gamma_2, \gamma_3, \gamma_4 < 1$ such that

$$\frac{\alpha}{2\gamma_1} + \frac{(p+n-2)-\frac{p}{2}\alpha}{(p+n-1)\gamma_2} = 1$$

and

$$\left(1+\frac{n-2}{p}\right)\frac{\alpha}{\gamma_3}+\frac{(p+n-2)(1-\alpha)}{(p+n-1)\gamma_4}=1.$$

In particular, we choose $\gamma_1=\frac{(p+n-1)\alpha}{2}$, $\gamma_2=\frac{(p+n-2)-\frac{p}{2}\alpha}{p+n-2}$, $\gamma_3=\frac{(p+n-2)(p+n-1)\alpha}{p}$ and $\gamma_4=(1-\alpha)$, apply the classical Young inequality to (3.22) and combine with (3.18) to obtain

$$\begin{aligned}&|\langle v(s)^{p+n-2}, Z^{(1)}(s)\rangle|\\ &\quad\lesssim \left(K(s)^{\gamma_1}+L(s)^{\gamma_2}+K(s)^{\gamma_3}+L(s)^{\gamma_4}+L(s)^{\frac{p+n-2}{p+n-1}}\right)\|Z^{(1)}(s)\|_{\mathcal{C}^{-\alpha}}.\end{aligned}$$

Using Young's inequality once more, now in the form

$$a\zeta^{\gamma}\le\gamma\frac{\zeta}{N^{\frac{1}{\gamma}}}+(1-\gamma)(Na)^{\frac{1}{1-\gamma}},$$

for $a=\|Z^{(1)}(s)\|_{\mathcal{C}^{-\alpha}}$, $\zeta\in\{K(s),L(s)\}$, $N=(Cn)^{\gamma}$ and $\gamma\in\{\gamma_1,\dots,\gamma_5\}$, where $\gamma_5=\frac{p+n-2}{p+n-1}$, we obtain the final bound

$$|\langle v(s)^{p+n-2}, Z^{(1)}(s)\rangle|\le\frac{1}{n}\left(K(s)+\frac{1}{2}L(s)\right)+C\sum_{i=1}^{5}\left(\|Z^{(1)}(s)\|_{\mathcal{C}^{-\alpha}}^{\frac{1}{1-\gamma_i}}\right), \tag{3.23}$$

for some $C>0$ which depends only on γ_i, $i\in\{1,2,\dots,5\}$, and n.

For the remaining terms in $\langle g(s), v(s)^{p-1}\rangle$ we need to estimate

$$\langle v(s)^{p+j-1}, Z^{(n-j)}(s)\rangle,$$

for all $0\le j\le n-2$. Proceeding in the same spirit of calculations as above we first obtain that

$$\|v(s)^{p+j-1}\|_{\mathcal{B}^{\alpha}_{1,1}}\lesssim K(s)^{\frac{\alpha}{2}}L(s)^{\frac{(p+j-1)-\frac{p}{2}\alpha}{p+n-1}}+K(s)^{\left(1+\frac{j-1}{p}\right)\alpha}L(s)^{\frac{(p+j-1)(1-\alpha)}{p+n-1}}+L(s)^{\frac{p+j-1}{p+n-1}}.$$

We define the exponents $\gamma_1^j=\frac{(p+n-1)\alpha}{2}$, $\gamma_2^j=\frac{(p+j-1)-\frac{p}{2}\alpha}{p+n-2}$, $\gamma_3^j=\frac{(p+j-1)(p+j)\alpha}{p}$ and $\gamma_4^j=\frac{(p+j)(1-\alpha)}{p+n-1}$. Note that (3.15) implies that $\gamma_1^j,\gamma_2^j,\gamma_3^j,\gamma_4^j<1$ and we also have that

$$\frac{\alpha}{2\gamma_1^j}+\frac{(p+j-1)-\frac{p}{2}\alpha}{(p+n-1)\gamma_2^j}=1$$

and

$$\left(1+\frac{j-1}{p}\right)\frac{\alpha}{\gamma_3^j}+\frac{(p+j-1)(1-\alpha)}{(p+n-1)\gamma_4^j}=1.$$

Applying Young's inequality once more

$$\begin{aligned}&|\langle v(s)^{p+j-1}, Z^{(n-j)}(s)\rangle| \\ &\quad\lesssim \left(K(s)^{\gamma_1^j} + L(s)^{\gamma_2^j} + K(s)^{\gamma_3^j} + L(s)^{\gamma_4^j} + L(s)^{\frac{p+j-1}{p+n-1}}\right) \|Z(s)^{(n-j)}\|_{\mathcal{C}^{-\alpha}}.\end{aligned}$$

As before (see (3.23)), we obtain the bound

$$|\langle v(s)^{p+j-1}, Z^{(n-j)}(s)\rangle| \leq \frac{1}{n}\left(K(s) + \frac{1}{2}L(s)\right) + C\sum_{i=1}^{5}\left(\|Z^{(n-j)}(s)\|_{\mathcal{C}^{-\alpha}}^{\frac{1}{1-\gamma_i^j}}\right), \tag{3.24}$$

for all $0 \leq j \leq n-2$, where $\gamma_5^j = \frac{p+j-1}{p+n-1}$. Thus, by (3.23) and (3.24),

$$|\langle g(s), v(s)^{p-1}\rangle| \leq \left(K(s) + \frac{1}{2}L(s)\right) + C\sum_{j=0}^{n-1}\sum_{i=1}^{5}\left(\|Z^{(n-j)}(s)\|_{\mathcal{C}^{-\alpha}}^{\frac{1}{1-\gamma_i^j}}\right), \tag{3.25}$$

where $\gamma_i^{n-1} = \gamma_i$, for all $i \in \{1, \ldots, 5\}$.

Finally, for $p_i^j = \frac{1}{1-\gamma_i^j}$, combining (3.16) and (3.25) we obtain

$$\frac{1}{p}\partial_s\|v(s)\|_{L^p}^p + \|v(s)\|_{L^p}^p + (p-2)K(s) + \frac{1}{2}L(s) \leq C\sum_{j,i}\|Z^{(n-j)}(s)\|_{\mathcal{C}^{-\alpha}}^{p_i^j}.$$

Let $t > s$ and notice that by (3.8), for $r \in (s,t)$,

$$\sum_{j,i}\|Z^{(n-j)}(r)\|_{\mathcal{C}^{-\alpha}}^{p_i^j} \leq \sum_{j,i} r^{-\alpha' p_i^j}\sup_{s\leq r\leq t}\left(r^{\alpha' p_i^j}\|Z^{(n-j)}(r)\|_{\mathcal{C}^{-\alpha}}^{p_i^j}\right)$$

for every $\alpha' > 0$. Thus for $r \in [s,t]$

$$\frac{1}{p}\partial_r\|v(r)\|_{L^p}^p + \frac{1}{2}L(r) \leq C\sum_{j,i} s^{-\alpha' p_i^j}\sup_{s\leq r\leq t}\left(r^{\alpha' p_i^j}\|Z^{(n-j)}(r)\|_{\mathcal{C}^{-\alpha}}^{p_i^j}\right).$$

By Jensen's inequality, for $\lambda = \frac{p+n-1}{p}$, we get that

$$\partial_r\|v(r)\|_{L^p}^p + C_1\left(\|v(r)\|_{L^p}^p\right)^\lambda \leq C_2\sum_{j,i} s^{-\alpha' p_i^j}\sup_{s\leq r\leq t}\left(r^{\alpha' p_i^j}\|Z^{(n-j)}(r)\|_{\mathcal{C}^{-\alpha}}^{p_i^j}\right),$$

and if we let $f(r) = \|v(r)\|_{L^p}^p$, $r \geq s$, by Lemma 3.11

$$\begin{aligned}f(r) \leq &\frac{f(s)}{\left(1 + (r-s)f(s)^{\lambda-1}(\lambda-1)\tilde{C}_1\right)^{\frac{1}{\lambda-1}}} \\ &\vee\left(\frac{2C_2}{C_1}\sum_{j,i} s^{-\alpha' p_i^j}\sup_{s\leq r\leq t}\left(r^{\alpha' p_i^j}\|Z^{(n-j)}(r)\|_{\mathcal{C}^{-\alpha}}^{p_i^j}\right)\right)^{\frac{1}{\lambda}},\end{aligned} \tag{3.26}$$

where $\tilde{C}_1 = C_1/2$. In particular for $r = t$ and $s = t/2$ we have the bound

$$\|v(t)\|_{L^p}^p \leq C \left[t^{-\frac{1}{\lambda-1}} \vee \left(\sum_{j,i} t^{-\alpha' p_i^j} \sup_{0 \leq r \leq t} \left(r^{\alpha' p_i^j} \|Z^{(n-j)}(r)\|_{\mathcal{C}^{-\alpha}}^{p_i^j} \right) \right)^{\frac{1}{\lambda}} \right],$$

which completes the proof. □

Lemma 3.11 (Comparison Test). *Let $\lambda > 1$ and $f : [0,T] \to [0,\infty)$ differentiable such that*

$$f'(t) + c_1 f(t)^\lambda \leq c_2,$$

for every $t \in [0,T]$. Then for $t > 0$

$$\begin{aligned} f(t) &\leq \frac{f(0)}{\left(1 + t f(0)^{\lambda-1}(\lambda-1)\frac{c_1}{2}\right)^{\frac{1}{\lambda-1}}} \vee \left(\frac{2c_2}{c_1}\right)^{\frac{1}{\lambda}} \\ &\leq t^{-\frac{1}{\lambda-1}} \left((\lambda-1)\frac{c_1}{2}\right)^{-\frac{1}{\lambda-1}} \vee \left(\frac{2c_2}{c_1}\right)^{\frac{1}{\lambda}}. \end{aligned}$$

Proof. Let $t > 0$. Then one of the following holds:

I. There exists $s_0 \leq t$ such that $f(s_0) \leq \left(\frac{2c_2}{c_1}\right)^{\frac{1}{\lambda}}$.

II. For every $s \leq t$, $f(s) > \left(\frac{2c_2}{c_1}\right)^{\frac{1}{\lambda}}$.

In the second case, using the assumption, we have that for every $s \leq t$

$$f'(s) + \frac{c_1}{2} f(s)^\lambda \leq 0$$

and solving the above differential inequality on $[0,t]$ implies that

$$f(t) \leq \frac{f(0)}{\left(1 + t f(0)^{\lambda-1}(\lambda-1)\frac{c_1}{2}\right)^{\frac{1}{\lambda-1}}}.$$

In the first case, assume for contradiction that $f(t) > \left(\frac{2c_2}{c_1}\right)^{\frac{1}{\lambda}}$ and let

$$s^* = \sup\left\{ s < t : f(s) \leq \left(\frac{2c_2}{c_1}\right)^{\frac{1}{\lambda}} \right\}.$$

Then $f(s) > \left(\frac{2c_2}{c_1}\right)^{\frac{1}{\lambda}}$, for every $s \in (s^*, t]$, while $f(s^*) = \left(\frac{2c_2}{c_1}\right)^{\frac{1}{\lambda}}$ by continuity. However, the assumption implies

$$f'(s) + \frac{c_1}{2} f(s)^\lambda \leq 0$$

and in particular $f'(s) \leq 0$. But then

$$f(t) = f(s^*) + \int_{s^*}^{t} f'(s)ds \leq \left(\frac{2c_2}{c_1}\right)^{\frac{1}{\lambda}},$$

which is a contradiction. □

3.4 Global Existence

The next theorem implies global existence of solutions to (3.3). Though it was already established in [MW17c], we present it here for completeness.

Theorem 3.12 ([MW17c, Theorem 6.1])**.** *For every initial condition* $x \in \mathcal{C}^{-\alpha_0}$ *and* $\beta > 0$ *as in* (3.5) *there exists a unique solution* $v \in C\left((0,\infty);\mathcal{C}^{\beta}\right)$ *of* (3.3).

Proof. Let $T > 0$. First fix any even integer $p \geq 2$ such that $L^p \hookrightarrow \mathcal{C}^{-\alpha_0}$ (for example $p \geq \frac{2}{\alpha_0}$ is enough; see also Proposition A.3 and (A.5)). Then the a priori estimate (3.14) (which depends only on $|||\underline{Z}|||_{\alpha;\alpha';T}$) provides an a priori estimate on $\|v(t)\|_{\mathcal{C}^{-\alpha_0}}$. Thus by Theorem 3.6 there exists $T_* \leq T$ bounded form below (by a constant depending only on the a priori estimate on $\|v(t)\|_{\mathcal{C}^{-\alpha_0}}$) and a unique solution up to time T_* of (3.3). Using again Theorem 3.6 we construct a solution of (3.3) on $[T_*, 2T_* \wedge T]$ with initial condition $v(T_*)$ which satisfies the same a priori bounds depending on $|||\underline{Z}|||_{\alpha;\alpha';T}$. We then proceed similarly until the whole interval $[0,T]$ is covered. To prove uniqueness we proceed as in the proof of Theorem [MW17c, Theorem 6.2]. □

3.5 Markov Property

For $x \in \mathcal{C}^{-\alpha_0}$ we write $X(\cdot;x) = \stackrel{\bullet}{|} + v$ where v is the solution to (3.2) with initial condition x. We introduce a variant of the notation (3.9) and set

$$\tilde{F}\left(v, \left(\stackrel{k}{\vee}\right)_{k=1}^{n}\right) = \sum_{k=0}^{n} a_k \sum_{j=0}^{k} \binom{k}{j} v^j \stackrel{k-j}{\vee}. \tag{3.27}$$

We denote by $B_b(\mathcal{C}^{-\alpha_0})$ the space of bounded functions and by $C_b(\mathcal{C}^{-\alpha_0})$ the space of bounded continuous functions from $\mathcal{C}^{-\alpha_0}$ to $\mathbb{R}$, both endowed with the norm

$$\|\Phi\|_\infty := \sup_{x\in\mathcal{C}^{-\alpha_0}} |\Phi(x)|.$$

For every $\Phi \in B_b(\mathcal{C}^{-\alpha_0})$ and $t \in [0,\infty)$ we define the map $P_t : \Phi \mapsto P_t\Phi$ by

$$P_t\Phi(x) := \mathbb{E}\Phi(X(t;x)), \tag{3.28}$$

for every $x \in \mathcal{C}^{-\alpha_0}$.

In this section we prove that $\{X(t;\cdot) : t \geq 0\}$ is a Markov process with transition semigroup $\{P_t : t \geq 0\}$ with respect to the filtration $\{\mathcal{F}_t : t \geq 0\}$ defined in (2.1).

We first prove the following lemma.

Lemma 3.13. *Let $X(\cdot;x) = \Psi + v$. Then, for every $h > 0$,*

$$X(t+h;x) = \Psi_t(t+h) + v_t(t+h),$$

where the remainder $v_t(t+\cdot)$ solves (3.2) *driven by the vector $\left(\Psi^{k}_t(t+\cdot)\right)_{k=1}^n$ and initial condition $X(t;x)$, that is,*

$$v_t(t+h) = S_1(h)X(t;x) - \int_0^h S_1(h-r)\tilde{F}\left(v_t(t+r), \left(\Psi^{k}_t(t+r)\right)_{k=1}^n\right) \mathrm{d}r.$$

Proof. Notice that for $h > 0$

$$X(t+h;x) = \Psi(t+h) + v(t+h) = \Psi_t(t+h) + v_t(t+h),$$

where

$$v_t(t+h) = S_1(h)X(t;x) - \int_0^h S_1(h-r)\tilde{F}\left(v(t+r), \left(\Psi^{k}(t+r)\right)_{k=1}^n\right) \mathrm{d}r.$$

By (2.11) we have that

$$\begin{aligned}\tilde{F}\left(v(t+r), \left(\Psi^{k}(t+r)\right)_{k=1}^n\right) &= \sum_{k=0}^n a_k \sum_{j=0}^k \binom{k}{j} v(t+r)^j \Psi^{k-j}(t+r)\\ &= \sum_{k=0}^n a_k \sum_{i=0}^k \binom{k}{i} v_t(t+r)^i \Psi^{k-i}_t(t+r),\end{aligned}$$

where we use a binomial expansion of $v(t+r)^j$ and a change of summation. Hence

$$\tilde{F}\left(v(t+r), \left(\Psi^{k}(t+r)\right)_{k=1}^n\right) = \tilde{F}\left(v_t(t+r), \left(\Psi^{k}_t(t+r)\right)_{k=1}^n\right),$$

which completes the proof. $\square$

The fact that $\{X(t;\cdot) : t \geq 0\}$ is a Markov process is immediate from the following theorem.

Theorem 3.14. *Let $X(\cdot;x)$ be as in the Lemma 3.13 with $x \in \mathcal{C}^{-\alpha_0}$. Then for every $\Phi \in B_b(\mathcal{C}^{-\alpha_0})$ and $t \geq 0$*

$$\mathbb{E}(\Phi(X(t+h;x))|\mathcal{F}_t) = P_h\Phi(X(t;x)),$$

for all $h \geq 0$.

Proof. Let $h \geq 0$ and $\Phi \in B_b(\mathcal{C}^{-\alpha_0})$ and write

$$\mathscr{T}\left(X(t;x); h; \left(\curlyvee^{k}_t(t+\cdot)\right)_{k=1}^n\right)$$

to denote the solution of (3.2) at time h, with $\left(\curlyvee^{k}\right)_{k=1}^n$ replaced by the vector $\left(\curlyvee^{k}_t(t+\cdot)\right)_{k=1}^n$ and initial condition $X(t;x)$. By Corollary 2.4 and [DPZ92, Proposition 1.12]

$$\mathbb{E}(\Phi(X(t+h;x))|\mathcal{F}_t) = \bar{\Phi}(X(t;x)),$$

where for $w \in \mathcal{C}^{-\alpha_0}$

$$\bar{\Phi}(w) = \mathbb{E}\Phi\left(\uparrow_t(t+h) + \mathscr{T}\left(w; h; \left(\curlyvee^{k}_t(t+\cdot)\right)_{k=1}^n\right)\right).$$

Here we use that $X(t;x)$ is $\mathcal{F}_t$-measurable and the vector $\left(\curlyvee^{k}_t(t+\cdot)\right)_{k=1}^n$ is independent of $\mathcal{F}_t$ (see Proposition 2.3). Given that $\left(\curlyvee^{k}_t(t+\cdot)\right)_{k=1}^n \overset{\text{law}}{=} \left(\curlyvee^{k}\right)_{k=1}^n$ (see again Proposition 2.3) and the fact that (3.2) has a unique solution driven by any vector $\underline{Y} \in C^{n,-\alpha}(0;T)$, for $T > 0$, and any initial condition $w \in \mathcal{C}^{-\alpha_0}$, we have that

$$\bar{\Phi}(w) = P_h\Phi(w),$$

which completes the proof if we set $w = X(t;x)$. $\square$

Theorem 3.14 implies that $\{P_t : t \geq 0\}$ is a semigroup. In the nect proposition we prove that it is Feller.

Proposition 3.15. *Let $\Phi \in C_b(\mathcal{C}^{-\alpha_0})$. Then, for every $t \geq 0$, $P_t\Phi \in C_b(\mathcal{C}^{-\alpha_0})$.*

Proof. It suffices to prove that the solution to (3.2) is continuous with respect to its initial condition. Fix $T > 0$ and $x \in \mathcal{C}^{-\alpha_0}$. Let $y \in \mathcal{C}^{-\alpha_0}$ such that $\|x - y\|_{\mathcal{C}^{-\alpha_0}} \leq 1$ and

$$v(t) = S_1(t)x - \int_0^t S_1(t-r)\tilde{F}\left(v(r), \left(\vee^{k}(r)\right)_{k=1}^{n}\right) \mathrm{d}r,$$
$$u(t) = S_1(t)y - \int_0^t S_1(t-r)\tilde{F}\left(u(r), \left(\vee^{k}(r)\right)_{k=1}^{n}\right) \mathrm{d}r,$$

as well as $\iota = \inf\{t > 0 : t^\gamma \|v(t) - u(t)\|_{\mathcal{C}^\beta} > 1\}$ and

$$M = \sup_{t \leq T} t^\gamma \|v_t\|_{\mathcal{C}^\beta},\ L = \left|\left|\left| \left(\vee^{k}_{0,\cdot}\right)_{k=1}^{n} \right|\right|\right|_{\alpha;\alpha';T}.$$

Notice that

$$\begin{aligned}
&\tilde{F}\left(v(r), \left(\vee^{k}(r)\right)_{k=1}^{n}\right) - \tilde{F}\left(u(r), \left(\vee^{k}(r)\right)_{k=1}^{n}\right) \\
&\quad = \sum_{k=0}^{n} a_k \sum_{j=0}^{k} \binom{k}{j} \left(u(r)^k - v(r)^k\right) \vee^{k-j}(r)
\end{aligned}$$

and by Propositions A.5, A.7 and A.8 we obtain that for all $T_* \leq T \wedge \iota$

$$\begin{aligned}
\sup_{t \leq T_*} t^\gamma \|v(t) - u(t)\|_{\mathcal{C}^\beta} &\leq \sup_{t \leq T_*} t^\gamma \|v(t) - u(t)\|_{\mathcal{C}^\beta} \sum_{m=1}^{n} \lambda_m T_*^{\alpha_m} \\
&\quad + \|x - y\|_{\mathcal{C}^{-\alpha_0}} \sum_{m=n+1}^{2n} \lambda_m T_*^{\alpha_m},
\end{aligned}$$

where $\lambda_m \equiv \lambda_m(M, L, \|x\|_{\mathcal{C}^{-\alpha_0}})$ and $\alpha_m \in (0,1]$. Choosing $T_* \equiv T_*(M, L, \|x\|_{\mathcal{C}^{-\alpha_0}})$ such that $T_* \leq 1/2$ we obtain

$$\sup_{t \leq T_*} t^\gamma \|v(t) - u(t)\|_{\mathcal{C}^\beta} \leq \|x - y\|_{\mathcal{C}^{-\alpha_0}}.$$

Iterating the procedure we find $M_* \geq 1$ and $C > 0$ such that

$$\sup_{t \leq T \wedge \iota} t^\gamma \|v(t) - u(t)\|_{\mathcal{C}^\beta} \leq (M_* C + 1)\|x - y\|_{\mathcal{C}^{-\alpha_0}},$$

for every $y \in \mathcal{C}^{-\alpha_0}$ such that $\|x - y\|_{\mathcal{C}^{-\alpha_0}} \leq 1$. At this point we should notice that for every $y \in \mathcal{C}^{-\alpha_0}$ such that $\|x - y\|_{\mathcal{C}^{-\alpha_0}} \leq 1/2(M_* C + 1)$ the above estimate implies that

$$\sup_{t \leq T \wedge \iota} t^\gamma \|v(t) - u(t)\|_{\mathcal{C}^\beta} \leq \frac{1}{2},$$

thus $T \wedge \iota = T$ because of the definition of ι. Hence, for all such $y \in \mathcal{C}^{-\alpha_0}$,

$$\sup_{t \leq T} t^{\gamma} \|v(t) - u(t)\|_{\mathcal{C}^{\beta}} \leq (M_* C + 1)\|x - y\|_{\mathcal{C}^{-\alpha_0}},$$

which implies convergence of $u(t)$ to $v(t)$ in $\mathcal{C}^{\beta}$ for every $0 < t \leq T$. Since T was arbitrary, the last implies continuity of the solution map of (3.2) with respect to its initial condition. The Feller property is then immediate if we combine the continuity of the solution map and the dominated convergence theorem. $\square$

Chapter 4

Ergodicity

4.1 Introduction

In this chapter we consider the stochastic quantisation equation (3.1). This equation was first proposed by Parisi and Wu in [PW81] as the natural reversible dynamics for the $\mathcal{P}(\varphi)_2$-Euclidean measure given by

$$\nu(\,\mathrm{d}X) \propto \exp\left\{-\int_{\mathbb{T}^2}\sum_{k=0}^{n}\frac{a_k}{k+1} : X^{k+1} : (z)\,\mathrm{d}z\right\}\mu(\,\mathrm{d}X), \tag{4.1}$$

where μ is the law of a massive Gaussian free field and $: X^{k+1} :$ stands for the $(k+1)$-th Wick power of X.

Motivated by Parisi and Wu's original proposal to construct and study the measure (4.1) as the equilibrium limit of the solutions to (3.1), our aim is to establish exponential convergence to a unique equilibrium.

Using the a priori estimates in Proposition 3.10 we first establish a strong dissipative bound for the solutions (see Theorem 4.1). We then prove the strong Feller property for the Markov semigroup generated by the solution generalising the method in [HSV07, Section 4.2] (see Theorem 4.13). Although for convenience we make (moderate) use of global in time existence which follows from Proposition 3.10, this part of the analysis could also be implemented using only local existence (see Remark 4.12); the linearised dynamics of Galerkin approximations are controlled by combining a localisation via stopping times and the small-time bounds obtained from the local existence theory. We furthermore establish a support theorem in the spirit of [CF16] (see Proposition 4.17

and Corollary 4.18). Finally, we combine all of these ingredients to show that the associated Markov semigroup satisfies the Doeblin criterion (see Theorem 4.19) which implies exponential convergence to the unique invariant measure uniformly over the state space (see Corollary 4.20).

All steps are implemented for general odd n except for the support theorem which we only show in the case $n = 3$. The reason for this restriction is explained in Remark 4.15. We expect however that a support theorem for (2.9) holds true for all odd n and that such a result could be combined with the results of this chapter to generalise Theorem 4.19 to the case of arbitrary odd n.

Along the way, as a corollary of the strong dissipative bound Theorem 4.1, we prove existence of invariant measures. A similar result was previously established in [RZZ17b] where the authors proved that (4.1) is a reversible (and in particular invariant) for (3.1), based on the identification of the dynamics with the solutions constructed via Dirichlet forms in [AR91]. We would like to point out that the approach presented here completely circumvents the theory of Dirichlet forms and uses neither the symmetry of the process nor the explicit form of the invariant measure. We therefore expect that our methods could be applied in situations where the reversibility is absent and where there is no explicit representation of the invariant measure, for example in situations where X is vector rather than scalar valued.

We would also like to mention two independent works on a similar subject that appeared around the same time with the results discussed here: [RZZ17a] and [HM18b]. In [RZZ17a] the authors established that (4.1) is the unique invariant measure for the dynamics and that the transition probabilities converge to this invariant measure. Their method was based on the asymptotic coupling technique from [HMS11] and relies on the bounds from [MW17c]. This analysis does however not include the strong Feller property or the support theorem and does not imply exponential convergence to equilibrium. In [HM18b] the authors presented a general method to establish the strong Feller property, for solutions to singular SPDEs solved in the framework of the theory of Regularity Structures [Hai14]. As an example, this method is implemented for the dynamic Φ^4_3. We expect that their method can also treat the case of (3.1) but at first glance it only implies continuity of the associated Markov semigroup with respect to the total variation distance, whereas Theorem 4.13 implies Hölder continuity.

4.1.1 Outline

In Section 4.2 we prove a strong dissipative bound for the solution to (3.1) as defined in Definition 3.3 and as a corollary we obtain existence of invariant measures. In Section 4.3 we prove the strong Feller property for the associated Markov semigroup. In Section 4.4 we prove a support theorem for the law of the solutions. Finally, in Section 4.5 we combine these results to prove exponential mixing of the law of the solutions.

4.1.2 Notation

Following Definition 3.3, we write $X(\cdot;x) = \upharpoonleft + v$ for the solution to (3.1) with initial condition x, where $\upharpoonleft$ is as in (2.3) for $s=0$ and the remainder term v solves (3.2) starting at x.

We fix α_0, β and γ as in (3.5) to measure the regularity of the initial condition x in $\mathcal{C}^{-\alpha_0}$, the regularity of v in $\mathcal{C}^{\beta}$ and the rate of blow-up of $\|v(t)\|_{\mathcal{C}^\beta}$ for t close to 0 (see Theorem 3.6).

We denote by $\underline{Z}$ the vector $\left(Z^{(j)}\right)_{j=0}^n$ for $Z^{(j)}$ as in (3.4) unless otherwise stated. We also denote by $C^{n,-\alpha}(0;T)$ the space defined in (3.6) on which we fix the norm $|\!|\!|\cdot|\!|\!|_{\alpha;\alpha';T}$ as in (3.7) for $\alpha, \alpha' > 0$ sufficiently small as in Chapter 3.

4.2 A Strong Dissipative Bound

We first prove the following corollary of Proposition 3.10 which implies a strong dissipative bound on the moments of X uniformly in time and the initial condition.

Theorem 4.1. *For $x \in \mathcal{C}^{-\alpha_0}$ let $X(\cdot;x)$ be the solution to* (3.1) *with initial condition x. Then for every $\alpha > 0$ and $p \geq 2$*

$$\sup_{x\in\mathcal{C}^{-\alpha_0}} \sup_{t\geq 0} \left(t^{\frac{p}{n-1}} \wedge 1\right) \mathbb{E}\|X(t;x)\|^p_{\mathcal{C}^{-\alpha}} < \infty. \tag{4.2}$$

Remark 4.2. Notice that the bound (4.2) does not follow immediately by taking expectations in (3.14). In fact the expectation of the supremum

$$\sup_{0\leq r\leq t} \left(r^{\alpha' p_i^j} \|Z^{(n-j)}(s)\|^{p_i^j}_{\mathcal{C}^{-\alpha}}\right)$$

on the right hand side of this estimate is finite for every $t < \infty$ but it is not uniformly bounded in t. However, as (3.14) does not depend on the initial condition we can just restart (3.1) at time $t-1$ for $t > 1$ and apply Proposition 3.10 for the restarted solution to obtain a bound which depends only on the randomness inside the interval $[t-1, t]$. Given that for every $j \geq 1$ the restarted stochastic objects $\overset{j}{\vee}_{\cdot}$ have the same law on intervals of the same size (see Proposition 2.3) we then obtain a bound which is independent of t.

Proof. Let $t > 1$ and notice that by Lemma 3.13 $X(t; x) = \upharpoonleft_{t-1}(t) + v_{t-1}(t)$ where $v_{t-1}(r)$, $r \geq t-1$, solves (3.2) with initial condition $X(t-1; x)$ and

$$Z^{(n-j)} = \sum_{k=j}^{n} a_k \binom{k}{j} \overset{k}{\vee}_{t-1}(t-1+\cdot),$$

for every $0 \leq j \leq n-1$. Applying Proposition 3.10 on v_{t-1} we then have

$$\|v_{t-1}(t)\|_{L^p}^p \lesssim 1 \vee \left(\sum_{j,i} \sup_{t-1 \leq r \leq t} \left(\left(r - (t-1)\right)^{\alpha' p_i^j} \|Z^{(n-j)}(r)\|_{\mathcal{C}^{-\alpha}}^{p_i^j} \right) \right)^{\frac{1}{\lambda}}, \tag{4.3}$$

for every $p \geq 2$. To prove (4.2) we fix $\alpha > 0$ and, using the embedding $L^p \hookrightarrow \mathcal{C}^{-\alpha}$ for $p \geq \frac{2}{\alpha}$ (see (A.5) and Proposition A.3), we first notice that for $t > 1$

$$\begin{aligned} &\mathbb{E}\|X(t;x)\|_{\mathcal{C}^{-\alpha}}^p \\ &\quad \lesssim \mathbb{E}\|\upharpoonleft_{t-1}(t)\|_{\mathcal{C}^{-\alpha}}^p + \mathbb{E}\|v_{t-1}(t)\|_{\mathcal{C}^{-\alpha}}^p \lesssim \mathbb{E}\|\upharpoonleft_{t-1}(t)\|_{\mathcal{C}^{-\alpha}}^p + \mathbb{E}\|v_{t-1}(t)\|_{L^p}^p. \end{aligned}$$

Combining with (4.3) and given that for every $k \geq 1$ the law of $\overset{k}{\vee}_{t-1}(t+\cdot)$ does not depend on t we obtain that

$$\sup_{t \geq 1} \mathbb{E}\|X(t;x)\|_{\mathcal{C}^{-\alpha}}^p < \infty.$$

Finally, using (3.14) (and by possibly tuning down α' in the same equation) for $t \leq 1$ we get

$$\mathbb{E}\|X(t;x)\|_{\mathcal{C}^{-\alpha}}^p \lesssim \mathbb{E}\|\upharpoonleft(t)\|_{\mathcal{C}^{-\alpha}}^p + \mathbb{E}\|v(t)\|_{L^p}^p \lesssim 1 + t^{-\frac{p}{n-1}},$$

which completes the proof. □

We denote by $\{P_t^* : t \geq 0\}$ the dual semigroup of $\{P_t : t \geq 0\}$ acting on the set of all probability Borel measures on $\mathcal{C}^{-\alpha_0}$ denoted by $\mathcal{M}_1(\mathcal{C}^{-\alpha_0})$. In the next corollary we prove existence of invariant measures of $\{P_t : t \geq 0\}$ as a semigroup acting on the set $C_b(\mathcal{C}^{-\alpha_0})$ of bounded continuous functions from $\mathcal{C}^{-\alpha_0}$ to $\mathbb{R}$ using Theorem 4.1.

Corollary 4.3. *For every $x \in \mathcal{C}^{-\alpha_0}$ there exists a measure $\mu_x \in \mathcal{M}_1(\mathcal{C}^{-\alpha_0})$ and a sequence $t_k \nearrow \infty$ such that*

$$\frac{1}{t_k}\int_0^{t_k} P_s^*\delta_x \,\mathrm{d}s \overset{weakly}{\longrightarrow} \mu_x.$$

In particular the measure μ_x is invariant for the Markov semigroup $\{P_t : t \geq 0\}$ on $\mathcal{C}^{-\alpha_0}$.

Proof. For $t > 0$ and $\alpha > 0$ using Markov's and Jensen's inequality there exists a constant $C > 0$ such that

$$\mathbb{P}(\|X(t;x)\|_{\mathcal{C}^{-\alpha}} > K) \leq \frac{C}{K}\left(\mathbb{E}\|X(t;x)\|^p_{\mathcal{C}^{-\alpha}}\right)^{\frac{1}{p}},$$

for every $K > 0$ and $p \geq 2$. Thus

$$\begin{aligned}\int_0^t \mathbb{P}(\|X(s;x)\|_{\mathcal{C}^{-\alpha}} > K)\,\mathrm{d}s &\leq \frac{C}{K}\int_0^t \left(\mathbb{E}\|X(s;x)\|^p_{\mathcal{C}^{-\alpha}}\right)^{\frac{1}{p}}\,\mathrm{d}s \\ &\leq \frac{C}{K}\left[\int_0^1 s^{-\frac{1}{n-1}}\,\mathrm{d}s + \int_1^t \mathrm{d}s\right] \\ &\leq \frac{C}{K}t\end{aligned}$$

where in the second inequality we use (4.2). If we let $R_t = \frac{1}{t}\int_0^t P_s^*\delta_x\,\mathrm{d}s$, for $K_\varepsilon = \frac{C}{\varepsilon}$ we get

$$R_t(\{f \in \mathcal{C}^{-\alpha} : \|f\|_{\mathcal{C}^{-\alpha}} > K_\varepsilon\}) \leq \varepsilon.$$

Choosing $\alpha < \alpha_0$ we can ensure that $\{f \in \mathcal{C}^{-\alpha} : \|f\|_{\mathcal{C}^{-\alpha}} \leq K_\varepsilon\}$ is a compact subset of $\mathcal{C}^{-\alpha_0}$ since the embedding $\mathcal{C}^{-\alpha} \hookrightarrow \mathcal{C}^{-\alpha_0}$ is compact for every $\alpha < \alpha_0$ (see Proposition A.4 and (A.2)). This implies tightness of $\{R_t\}_{t\geq 0}$ in $\mathcal{C}^{-\alpha_0}$ and by the Krylov–Bogoliubov existence Theorem (see [DPZ96, Corollary 3.1.2]) there exist a sequence $t_k \nearrow \infty$ and a measure $\mu_x \in \mathcal{M}_1(\mathcal{C}^{-\alpha_0})$ such that $R_{t_k} \to \mu_x$ weakly in $\mathcal{C}^{-\alpha_0}$ and μ_x is invariant for the semigroup $\{P_t : t \geq 0\}$ in $\mathcal{C}^{-\alpha_0}$. □

4.3 The Strong Feller Property

In this section we show that the Markov semigroup $\{P_t : t \geq 0\}$ satisfies the strong Feller property. The strong Feller property is to be expected when we deal with SPDEs where the noise forces every direction in Fourier space. However, the fact that the process X does not solve a self-contained equation forces us to translate everything onto the level of the remainder v. The most important step is to obtain a Bismut–Elworthy–Li formula (see Theorem 4.8) which captures enough information to provide a good control of the linearisation of the remainder equation.

On the technical level, we work with a finite dimensional approximation X^N of X. This choice and the fact that the equation is driven by white noise imply that the solution is Fréchet differentiable with respect to the (finite dimensional approximation of the) noise, so we can avoid working with Malliavin derivatives. This is expressed in Proposition 4.4 below, and in fact this proposition could even be established without splitting X^N into v^N and Ψ^N. We make strong use of the splitting in Proposition 4.7 where the local solution theory is used to obtain deterministic bounds on v^N and its linearisation for small t provided that we control the diagrams $\vee_n^N$. This control is uniform in N and enters crucially the proof of Proposition 4.11.

From now on we fix $\alpha \in (0, \alpha_0)$ sufficiently small. For $N \geq 1$ let $\Pi_N L^2$ be the finite dimensional subspace of L^2 spanned by $\{e_m\}_{|m|<N}$ (recall that we deal with real-valued functions and the symmetry condition (1.11) is always valid) and denote by Π_N the corresponding orthogonal projection. We also let $\hat{\Pi}_N$ be a linear smooth approximation taking values in $\Pi_N L^2$ and having the properties i and ii introduced in the discussion before Proposition 3.7.

Let $\Re_N$ be the renormalisation constant defined in (2.9) and consider a finite dimensional approximation of (3.1) given by

$$
\begin{aligned}
\mathrm{d}X^N(t) &= \left((\Delta - 1)X^N(t) - \sum_{k=0}^{n} a_k \hat{\Pi}_N \mathcal{H}_k \left(X^N(t), \Re_N \right) \right) \mathrm{d}t + \mathrm{d}W_N(t, \cdot) \\
X^N|_{t=0} &= \hat{\Pi}_N x
\end{aligned}
\tag{4.4}
$$

for some initial condition $x \in \mathcal{C}^{-\alpha_0}$. Here $W_N(t, z) = \sum_{|m|<N} \hat{W}_m(t) e_m(z)$, where $(\hat{W}_m)_{m \in \mathbb{Z}^2}$ is a family of complex Brownian motions such that $\hat{W}_{-m} = \overline{\hat{W}_m}$ and independent otherwise. We furthermore assume that W_N is defined on the same probability space Ω as ξ via the identity

$$
\hat{W}_m(t) := \sqrt{2}\xi \left(\mathbf{1}_{[0,t]} \times e_m \right), \quad m \in \mathbb{Z}^2,
$$

which also makes it adapted with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$. It is convenient to write $W_N = G_N(\hat{W}_m)_{m \in \mathbb{Z}^2 \cap (-N,N)^2}$ for $G_N : C\left([0, \infty); \mathbb{R}^{(2N-1)^2} \right) \to$

$C\left([0,\infty);\Pi_N L^2\right)$ such that

$$G_N(\hat{W}_m)_{m\in\mathbb{Z}^2\cap(-N,N)^2} = \sum_{|m|<N} \hat{W}_m e_m.$$

The Cameron–Martin space of W_N is given by

$$\mathcal{CM} := W_0^{1,2}([0,\infty)) = \left\{w : \partial_t w \in L^2\left([0,\infty);\mathbb{R}^{(2N-1)^2}\right),\, w(0)=0\right\}.$$

Last, we have the identity

$$\upharpoonleft^N(t) = \sum_{|m|<N} \int_0^t \mathrm{e}^{-(1+4\pi^2|m|^2)(t-s)}\, \mathrm{d}\hat{W}_m(s)\, e_m, \tag{4.5}$$

where $\upharpoonleft^N$ is the finite dimensional approximation defined in Section 2.2.

For $v \in \mathcal{C}^\beta$ and $\underline{Z} \in (\mathcal{C}^{-\alpha})^n$, $\alpha < \beta$, we use the notation

$$\tilde{F}(v,\underline{Z}) = \sum_{k=0}^n a_k \sum_{j=0}^k \binom{k}{j} v^j Z^{(k-j)}$$

with the convention that $Z^{(0)} \equiv 1$. Recall that $\tilde{F}$ (see (3.27)) is a variant of F in (3.9). Here and for the rest of this section $\underline{Z}$ is a shortcut for $\left(\overset{k}{\vee}\right)_{k=1}^n$ (notice that this differs from the convention used in Chapter 3). We also let

$$\tilde{F}'(v,\underline{Z}) = \sum_{k=1}^n k a_k \sum_{j=0}^{k-1} \binom{k-1}{j} v^j Z^{(k-1-j)}.$$

Formally, $\tilde{F}'$ stands for the derivative of $\sum_{k=0}^n a_k : X^k :$ with respect to X, with $: X^k :$ replaced by $\sum_{j=0}^k \binom{k}{j} v^j Z^{(k-j)}$.

From now on we furthermore denote by $\mathcal{D}$ the Fréchet derivative with respect to elements in $C\left([0,t];\mathbb{R}^{(2N-1)^2}\right)$ (i.e. with respect to the noise), for $t>0$, and by D the Fréchet derivative with respect to elements in $\mathcal{C}^{-\alpha_0}$ (i.e. with respect to the initial condition).

Existence and uniqueness of local in time solutions to (4.4) up to some random explosion time $T_*^N > 0$ can be proven following the same method as in Section 3, i.e. using the ansatz $X^N = \upharpoonleft^N + v^N$ and solving the PDE problem

$$\begin{aligned}&\big(\partial_t - (\Delta - 1)\big)v^N = -\hat{\Pi}_N \tilde{F}(v^N, \underline{Z}^N)\\ &v^N|_{t=0} = \hat{\Pi}_N x\end{aligned}, \tag{4.6}$$

where $\underline{Z}^N = \left(\overset{k}{\vee}^N\right)_{k=1}^n$ (see Section 2.2 for definitions).

Notice that for fixed v, $\tilde{F}$ is Fréchet differentiable with respect to any $\underline{Z} \in (\mathcal{C}^{-\alpha})^n$ as a function taking values in $\mathcal{C}^{-\alpha}$. Recall that $\overset{k}{\vee}^N = \mathcal{H}_k(\Psi^N, \Re_N)$, for every $1 \leq k \leq n$, so that the map

$$\left(v, Z^{(1)}\right) \mapsto S_1(t)\hat{\Pi}_N x - \int_0^t S_1(t-s)\hat{\Pi}_N \tilde{F}\left(v(s), \left(\mathcal{H}_k\left(Z^{(1)}(s), \Re_N\right)\right)_{k=1}^n\right) \mathrm{d}s, \quad (4.7)$$

for $\left(v, Z^{(1)}\right) \in C\left([0,t]; \mathcal{C}^\beta\right) \times C\left([0,t]; \Pi_N L^2\right)$ and $t > 0$, is Fréchet differentiable as a composition of $\tilde{F}$ with a linear operator shifted by a constant, since the mapping

$$C\left([0,t]; \Pi_N L^2\right) \ni Z^{(1)} \mapsto \left(\mathcal{H}_k\left(Z^{(1)}, \Re_N\right)\right)_{k=1}^n \in C^{n,-\alpha}(0;t)$$

is Fréchet differentiable for any $\alpha > 0$, with respect to any $|||\cdot|||_{\alpha;\alpha';t}$, for $\alpha' > 0$ fixed. Thus, for fixed $x \in \mathcal{C}^{-\alpha_0}$ and $\Psi^N \in C\left([0,t]; \Pi_N L^2\right)$ the implicit function theorem for Banach spaces (see [Zei95, Theorem 4E]) can be applied up to time $T_*^N \equiv T_*^N(x, \Psi^N)$ where existence of v^N is ensured. Hence, for $t \in (0, T_*^N)$ there exists an open neighbourhood $\mathcal{U}_{\Psi^N} \subset C\left([0,t]; \Pi_N L^2\right)$ of Ψ^N such that the solution map $\mathscr{T}_t^{N,x} : \mathcal{U}_{\Psi^N} \to \mathcal{C}^\beta$ of (4.6) is Fréchet differentiable at Ψ^N.

Using Itô's formula the stochastic integrals in (4.5) can be written as

$$\begin{aligned} &\int_0^\cdot \mathrm{e}^{-(1+4\pi^2|m|^2)(\cdot - s)} \,\mathrm{d}\hat{W}_m(s) \\ &\quad = \hat{W}_m(\cdot) - (1 + 4\pi^2|m|^2) \int_0^\cdot \mathrm{e}^{-(1+4\pi^2|m|^2)(\cdot - s)} \hat{W}_m(s) \,\mathrm{d}s. \end{aligned} \quad (4.8)$$

Notice that the right hand side in the above equation is well-defined if we replace $(\hat{W}_m)_{m \in \mathbb{Z}^2 \cap (-N,N)^2}$ by any $w \in C\left([0,t]; \mathbb{R}^{(2N-1)^2}\right)$, therefore (4.8) is a continuous linear function on $C([0,t]; \mathbb{R}^{(2N-1)^2})$. Thus Ψ^N as a function from $C\left([0,t]; \mathbb{R}^{(2N-1)^2}\right)$ to $C\left([0,t]; \Pi_N L^2\right)$ is Fréchet differentiable. Combining all the above we finally obtain Fréchet differentiability of $v(t)^N$ from $C\left([0,t]; \mathbb{R}^{(2N-1)^2}\right)$ to $\mathcal{C}^\beta$.

We let $\hat{W}_N = (\hat{W}_m)_{|m|<N}$ and for $w \in C\left([0,t]; \mathbb{R}^{(2N-1)^2}\right)$ we write

$$\int_0^t S_1(t-s) G_N \,\mathrm{d}w(s) := \sum_{|m|<N} \int_0^t \mathrm{e}^{-(1+4\pi^2|m|^2)(t-s)} \,\mathrm{d}w_m(s),$$

where the right hand side is defined as in (4.8) with $\hat{W}_m$ replaced by w_m.

In the next proposition we summarise the results of the previous discussion.

Proposition 4.4. *For $x \in \mathcal{C}^{-\alpha_0}$, $\hat{W}_N \in C([0,\infty);\mathbb{R}^{(2N-1)^2})$ and $\upharpoonleft^N \equiv \upharpoonleft^N(\hat{W}_N) \in C([0,\infty);\Pi_N L^2)$, let $T_*^N \equiv T_*^N(x, \upharpoonleft^N) > 0$ be the explosion time of v^N. Then for all $t < T_*^N$ there exists an open neighbourhood $\mathcal{O}_{\hat{W}_N} \subset C\left([0,t];\mathbb{R}^{(2N-1)^2}\right)$ of $\hat{W}_N$ such that $X^N(t;x)(= \upharpoonleft^N(t) + v^N(t))$ is Fréchet differentiable as a function from $\mathcal{O}_{\hat{W}_N}$ to $\mathcal{C}^{-\alpha_0}$ and for any $w \in C\left([0,t];\mathbb{R}^{(2N-1)^2}\right)$ its directional derivative $\mathcal{D}X^N(t;x)(w)$ is given in mild form as*

$$\begin{aligned}\mathcal{D}X^N(t;x)(w) = &-\int_0^t S_1(t-s)\hat{\Pi}_N\left[\tilde{F}'(v^N(s),\underline{Z}^N(s))\mathcal{D}X^N(s;x)(w)\right]\mathrm{d}s \\ &+\int_0^t S_1(t-s)G_N\,\mathrm{d}w(s). \end{aligned}\tag{4.9}$$

Remark 4.5. We expect that $T_*^N = \infty$, which we already established in the limiting case $N \to \infty$ in Section 3. However, here we only use the local solution theory to control the semigroup associated to $X^N(t;x)$ (see Proposition 4.11), thus we do not insist on proving a global existence theorem. We then pass to the limit using the fact that $T_*^N \to \infty$ (see the discussion above Remark 4.12).

Proof. The Fréchet differentiability of $X^N(t;x)$ follows by the previous discussion and (4.9) by differentiating (4.7). □

For $h \in \mathcal{C}^{-\alpha_0}$, we let $h_N = \hat{\Pi}_N h$ and for $t \geq s$ we also consider the following linear equation,

$$\begin{aligned}&\big(\partial_t - (\Delta - 1)\big)J_{s,t}^N h_N = -\hat{\Pi}_N\left[\tilde{F}'(v^N(t),\underline{Z}^N(t))J_{s,t}^N h_N\right] \\ &J_{s,t}^N h_N|_{t=s} = h_N\end{aligned}.\tag{4.10}$$

Then $J_{0,t}^N h_N = DX^N(t;x)(h)$, i.e. it is the derivative of $X^N(t;\cdot)$ in the direction h, and its existence for every $t \leq T_*^N$ is ensured by a similar argument as the one discussed before Proposition 4.4.

At this point we should comment on the relation between (4.9) and (4.10). Given that (4.10) has a unique solution for every $h_N \in \Pi_N L^2$ up to time $t > 0$, then for $w \in \mathcal{CM}$, i.e. $w(0) = 0$ and $\partial_t w \in L^2\left([0,\infty);\mathbb{R}^{(2N-1)^2}\right)$, by Duhamel's principle

$$\mathcal{D}X^N(t;x)(w) = \int_0^t J_{s,t}^N G_N \partial_s w(s)\,\mathrm{d}s,\tag{4.11}$$

where $J_{s,t}^N : \mathcal{C}^{-\alpha_0} \to \mathcal{C}^\beta$ is the solution map of (4.10).

Remark 4.6. In the framework of Malliavin calculus $\mathcal{D}_s X^N(t;x) = J^N_{s,t} G_N$ as an element of the dual of $L^2\left([0,\infty);\mathbb{R}^{(2N-1)^2}\right)$ is the Malliavin derivative (see [Nua06, Section 1.2]) in the sense that the latter coincides with the former when it acts on $X^N(t;x)$. In our case, the presence of additive noise implies Fréchet differentiability with respect to the noise as an element in $C\left([0,t];\mathbb{R}^{(2N-1)^2}\right)$ (see Proposition 4.4), which is stronger than Malliavin differentiability with respect to the noise.

For $r \in \left[\frac{1}{4}, 1\right]$ which we fix below and $0 < \alpha' < \alpha$ we consider the stopping times

$$\begin{aligned} \tau_r^N &:= \inf\left\{ t > 0 : \|\Psi^N(t)\|_{\mathcal{C}^{-\alpha}} \vee \ldots \vee t^{\alpha'(n-1)} \|\Psi^{n,N}(t)\|_{\mathcal{C}^{-\alpha}} > r \right\}. \\ \tau_r &:= \inf\left\{ t > 0 : \|\Psi(t)\|_{\mathcal{C}^{-\alpha}} \vee \ldots \vee t^{(n-1)\alpha'} \|\Psi^{n}(t)\|_{\mathcal{C}^{-\alpha}} > r \right\}. \end{aligned} \tag{4.12}$$

Let $\bar{B}(x;1)$ be the closed unit ball centred at x in $\mathcal{C}^{-\alpha_0}$. The next proposition provides local bounds on v^N and $J^N_{0,\cdot}$ given deterministic control on $\underline{Z}^N$ (see also Theorem 3.6).

Proposition 4.7. *Let $x \in \mathcal{C}^{-\alpha_0}$ and $R = 2\|x\|_{\mathcal{C}^{-\alpha_0}} + 1$. Then there exists a deterministic time $T_* \equiv T_*(R) > 0$, independent of N, such that for every $t \leq T_* \wedge \tau_r^N$ and initial conditions $y \in \bar{B}(x;1)$,*

$$\sup_{s \leq t} s^\gamma \|v^N(s)\|_{\mathcal{C}^\beta} \leq 1 \textit{ and } \sup_{s\leq t} s^\gamma \|J^N_{0,s} h_N\|_{\mathcal{C}^\beta} \leq 2\|h_N\|_{\mathcal{C}^{-\alpha_0}},$$

for β, γ as in (3.5)*, uniformly in N, for every $h_N \in \hat{\Pi}_N L^2$.*

Proof. Let $t \leq \tau_r^N \wedge T_*$ where T_* is defined as in (3.10). We can also assume that $t \leq 1$. Then, from Theorem 3.6, we have that

$$\sup_{s\leq t} s^\gamma \|v^N(s)\|_{\mathcal{C}^\beta} \leq 1,$$

for every $y \in \bar{B}(x;1)$. Using Proposition A.5, (A.3) and (A.4) we get that

$$\begin{aligned} &\|S_1(t-s)\hat{\Pi}_N[\tilde{F}'(v^N(s), \underline{Z}^N(s))J^N_{0,s}h_N]\|_{\mathcal{C}^\beta} \\ &\quad \lesssim \left(s^{-(n-1)\gamma} + (t-s)^{-\frac{\beta+\alpha}{2}} s^{-(n-2)\gamma}\right) \|J^N_{0,s}h_N\|_{\mathcal{C}^\beta}, \end{aligned} \tag{4.13}$$

where we also use the fact that $\|\hat{\Pi}_N f\|_{\mathcal{C}^{-\alpha}} \lesssim \|f\|_{\mathcal{C}^{-\alpha}}$, for every $f \in \mathcal{C}^{-\alpha}$. We are now ready to retrieve the appropriate bounds on the operator norm of $J^N_{0,\cdot}$. For $h_N \in \Pi_N L^2$ we have in mild form,

$$J^N_{0,t} h_N = S_1(t) h_N - \int_0^t S_1(t-s)\hat{\Pi}_N\left[\tilde{F}'(v^N(s), \underline{Z}^N(s))J^N_{0,s}h_N\right] \mathrm{d}s.$$

Thus for $s \le t \le \tau_r^N \wedge T_*$ and $\alpha > 0$ sufficiently small (to ensure integrability of powers of s and $t-s$; see also (3.5)) by (4.13)

$$\|J_{0,s}^N h_N\|_{\mathcal{C}^\beta} \le Cs^{-\frac{\beta+\alpha_0}{2}}\|h_N\|_{\mathcal{C}^{-\alpha_0}} + C\left(s^{1-n\gamma} + s^{1-\frac{\beta+\alpha}{2}-(n-1)\gamma}\right)\sup_{s\le t} s^\gamma \|J_{0,s}^N h_N\|_{\mathcal{C}^\beta}.$$

Multiplying the above inequality by s^γ we get

$$\sup_{s\le t} s^\gamma \|J_{0,s}^N h_N\|_{\mathcal{C}^\beta} \le Ct^{\gamma-\frac{\beta+\alpha_0}{2}}\|h_N\|_{\mathcal{C}^{-\alpha_0}} + Ct^\theta \sup_{s\le t} s^\gamma \|J_{0,s}^N h_N\|_{\mathcal{C}^\beta},$$

for some $\theta \equiv \theta(\alpha,\beta,\gamma,n) > 0$. Using that $\gamma - \frac{\beta+\alpha_0}{2} > 0$ (see (3.5)) and by possibly changing the value of the constant C in (3.10) we finally obtain the bound

$$\sup_{s\le t} s^\gamma \|J_{0,s}^N h_N\|_{\mathcal{C}^\beta} \le 2\|h_N\|_{\mathcal{C}^{-\alpha_0}}, \tag{4.14}$$

which completes the proof. $\square$

We denote by $C_b^1(\mathcal{C}^{-\alpha_0})$ the set of continuously differentiable functions on $\mathcal{C}^{-\alpha_0}$. We furthermore let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ such that $\chi(\zeta) \in [0,1]$, for every $\zeta \in \mathbb{R}$, and

$$\chi(\zeta) = \begin{cases} 1, & \text{if } |\zeta| \le \frac{r}{2} \\ 0, & \text{if } |\zeta| \ge r \end{cases},$$

for r as in (4.12). For simplicity we also write $|||\cdot|||_t$ instead of $|||\cdot|||_{\alpha;\alpha';t}$. Inspired by [Nor86], we prove the following version of the Bismut–Elworthy–Li formula.

Theorem 4.8 (Bismut–Elworthy–Li formula)**.** *Let $x \in \mathcal{C}^{-\alpha_0}$, $\Phi \in C_b^1(\mathcal{C}^{-\alpha_0})$ and let $t > 0$. Let w be a process taking values in the Cameron-Martin space $\mathcal{CM}$ with $\partial_s w$ adapted. Furthermore, assume that there exists a deterministic constant $C \equiv C(t) > 0$ such that $\|\partial_s w\|^2_{L^2\left([0,t];\mathbb{R}^{(2N-1)^2}\right)} \le C$ $\mathbb{P}$-almost surely. Then we have the identity*

$$\begin{aligned}
&\mathbb{E}\left(D\Phi(X^N(t;x))\left(\mathcal{D}X^N(t;x)(w)\right)\chi\left(|||\underline{Z}^N|||_t\right)\right) \\
&\quad = \mathbb{E}\left(\Phi(X^N(t;x))\int_0^t \partial_s w(s)\cdot \mathrm{d}\hat{W}_N(s)\,\chi\left(|||\underline{Z}^N|||_t\right)\right) \\
&\qquad - \mathbb{E}\left(\Phi(X^N(t;x))\,\partial_+\chi\left(|||\underline{Z}^N|||_t\right)(w)\right).
\end{aligned} \tag{4.15}$$

Here

$$\begin{aligned}
&\partial_+\chi\left(|||\underline{Z}^N|||_t\right)(w) \\
&\quad = \partial_\zeta\chi\left(|||\underline{Z}^N|||_t\right)\partial_+|||\underline{Z}^N|||_t\Big(Q_w(\cdot), 2\,{\stackrel{\bullet}{|}}^{N}Q_w(\cdot), \dots, n\,\overset{n-1}{\vee}^{N}Q_w(\cdot)\Big),
\end{aligned} \tag{4.16}$$

where $\partial_+ |||\cdot|||_t : C^{n,-\alpha}(0;t) \to C^{n,-\alpha}(0;t)^*$ *is the one-sided derivative of* $|||\cdot|||_t$ *given by*

$$\partial_+ |||\underline{Z}^N|||_t(\underline{Y}) = \lim_{\delta\to 0^+} \frac{|||\underline{Z}^N + \delta\underline{Y}|||_t - |||\underline{Z}^N|||_t}{\delta},$$

for every direction $\underline{Y} \in C^{n,-\alpha}(0;t)$*, and*

$$Q_w(\cdot) := \int_0^{\cdot} S_1(\cdot - s) G_N \partial_s w(s)\,\mathrm{d}s.$$

Remark 4.9. It is worth mentioning that the usual Bismut–Elworthy–Li formula (see [Nor86]) gives an explicit representation of derivatives with respect to the initial condition rather than the noise. In (4.22) we also prove such a representation. However the core of our argument is (4.15) which is slightly more general than (4.22).

Remark 4.10. The presence of $\partial_+ |||\cdot|||_t$ in the theorem above is based on the fact that norms are not in general Fréchet differentiable functions. However, their one-sided derivatives always exist (see [DPZ92, Appendix D]) and they behave nicely in terms of the usual rules of differentiation.

Proof. Let $\delta > 0$ and $u = \partial_t w$, which is an $L^2\left([0,\infty);\mathbb{R}^{(2N-1)^2}\right)$ function. For every $n \geq 1$, we define the shift $T_{\delta u}$ by

$$T_{\delta u} \overset{n}{\triangledown}{}^N(t) = \sum_{k=0}^{n} \binom{n}{k} \Big(\delta Q_w(t)\Big)^k \overset{n-k}{\triangledown}{}^N(t)$$

and we let $T_{\delta u}\underline{Z}^N = \left(T_{\delta u}\overset{k}{\triangledown}{}^N\right)_{k=1}^n$.

Let $X^{N,\delta}(\cdot;x) = T_{\delta u}\upharpoonleft^N_{0,\cdot} + v^{N,\delta}$, where the remainder $v^{N,\delta}$ solves the equation

$$\begin{aligned} &\big(\partial_t - (\Delta - 1)\big)v^{N,\delta} = -\hat{\Pi}_N \tilde{F}(v^{N,\delta}, T_{\delta u}\underline{Z}^N) \\ &v^{N,\delta}|_{t=0} = \hat{\Pi}_N x \end{aligned}.$$

As in [Nor86], our aim is to construct a probability measure $\mathbb{P}^\delta$ such that the law of $T_{\delta u}\upharpoonleft^N$ under $\mathbb{P}^\delta$ is the same as the law of $\upharpoonleft^N$ under $\mathbb{P}$. That way we obtain the identity

$$\partial_{\delta+}\mathbb{E}_{\mathbb{P}^\delta}\left(\Phi\left(X^{N,\delta}(t;x)\right)\chi\left(|||T_{\delta u}\underline{Z}^N|||_t\right)\right)\Big|_{\delta=0} = 0, \tag{4.17}$$

since $\overset{k}{\triangledown}{}^N$ is a continuous function of $\upharpoonleft^N$ for every $k \geq 2$, the solution map to (4.6) is a continuous function of the $\overset{k}{\triangledown}{}^N$, and χ is a continuous function of

$\upharpoonleft^N$. Above $\partial_{\delta+}$ stands as a shortcut of the directional derivative of a function as $\delta \to 0^+$. We will then show below that the result follows by an expansion of the derivative in the above expression.

We start with the construction of $\mathbb{P}^\delta$. Let $\mathsf{B}^\delta(r) := -\int_0^r \delta u(s) \cdot \mathrm{d}\hat{W}_N(s)$ where $\cdot$ is the scalar product on $\mathbb{R}^{(2N-1)^2}$, and define the exponential process

$$\mathsf{A}^\delta(r) := \exp\left\{\mathsf{B}^\delta(r) - \frac{1}{2}\int_0^r |\delta u(s)|^2 \,\mathrm{d}s\right\}.$$

Notice that by the assumptions on w Novikov's condition is satisfied, i.e.

$$\mathbb{E}\exp\left\{\frac{1}{2}\int_0^t |\delta u(s)|^2 \,\mathrm{d}s\right\} < \infty,$$

thus by [RY99, Chapter 8, Proposition 1.15] A^δ is a strictly positive martingale and we have that $\mathbb{E}\mathsf{A}^\delta(t) = 1$. We define $\mathbb{P}^\delta$ by its Radon–Nikodym derivative with respect to $\mathbb{P}$

$$\frac{\mathrm{d}\,\mathbb{P}^\delta}{\mathrm{d}\,\mathbb{P}} = \mathsf{A}^\delta(t).$$

By Girsanov's Theorem (see [RY99, Chapter 4, Theorem 1.4]) we have that $\hat{W}_N^\delta(r) := \hat{W}_N(r) - \left[\hat{W}_N(\cdot), \mathsf{B}^\delta(\cdot)\right]_r$, $r \le t$, under $\mathbb{P}^\delta$ has the same law as $\hat{W}_N$ under $\mathbb{P}$, where $[\,\cdot\,,\cdot\,]_r$ stands for the quadratic variation at time r. We furthermore have that $\left[\hat{W}_N(\cdot), \mathsf{B}^\delta(\cdot)\right]_r = -\int_0^r \delta u(s)\,\mathrm{d}s$ as well as $\upharpoonleft^N(t) = \int_0^t S_1(t-s)G_N\,\mathrm{d}\hat{W}_N(s)$ and $T_{\delta u}\upharpoonleft^N(t) = \int_0^t S_1(t-s)G_N\,\mathrm{d}\hat{W}_N^\delta(s)$. Since the law of $\hat{W}_N^\delta$ under $\mathbb{P}^\delta$ is the same as the law of $\hat{W}^N$ under $\mathbb{P}$, this is also the case for $T_{\delta u}\upharpoonleft^N$ and $\upharpoonleft^N$ (recall that $\upharpoonleft^N$ is a continuous function of $\hat{W}_N$, when the latter is seen as an element in $C\left([0,t];\mathbb{R}^{(2N-1)^2}\right)$ endowed with the supremum norm because of (4.8)). Thus $\mathbb{P}^\delta$ is the required measure and (4.17) in the form

$$\partial_{\delta+}\mathbb{E}\left(\Phi\left(X^{N,\delta}(t;x)\right)\chi\left(\left|\!\left|\!\left|T_{\delta u}\underline{Z}^N\right|\!\right|\!\right|_t\right)\mathsf{A}^\delta(t)\right)\Big|_{\delta=0} = 0 \tag{4.18}$$

follows. Using the chain rule, $\partial_\delta\Phi\left(X^{N,\delta}(x;t)\right) = D\Phi\left(X^{N,\delta}(x;t)\right)\left(\partial_\delta X^{N,\delta}(x;t)\right)$ and $\partial_\delta\mathsf{A}^\delta(t) = -\mathsf{A}^\delta(t)\left(\int_0^t u(s)\cdot\mathrm{d}\hat{W}_N(s) + \delta\int_0^t |u(s)|^2\,\mathrm{d}s\right)$. For the directional derivative of $\chi\left(\left|\!\left|\!\left|T_{\delta u}\underline{Z}^N\right|\!\right|\!\right|_t\right)$ at $\delta^+ = 0$ it suffices to check the existence of the limit

$$\lim_{\delta\to 0^+}\frac{\left|\!\left|\!\left|T_{\delta u}\underline{Z}^N\right|\!\right|\!\right|_t - \left|\!\left|\!\left|\underline{Z}^N\right|\!\right|\!\right|_t}{\delta}.$$

We claim that the above limit is the same as

$$\partial_+ \left\|\!\left|\underline{Z}^N\right|\!\right\|_t \left(\underline{Y}^N\right) := \lim_{\delta\to 0^+} \frac{\left\|\!\left|\underline{Z}^N + \delta \underline{Y}^N\right|\!\right\|_t - \left\|\!\left|\underline{Z}^N\right|\!\right\|_t}{\delta}$$

where $\underline{Y}^N = \Big(Q_w(\cdot), 2\,\dot{|}^{N}_{0,\cdot} Q_w(\cdot), \dots, n \overset{n-1}{\bullet\!\!\vee\!\!\bullet}{}^{N}_{0,\cdot} Q_w(\cdot)\Big)$. Using the fact that $\left\|\!\left|\cdot\right|\!\right\|_t$ is a norm, we have that

$$\frac{\left\|\!\left|T_{\delta u}\underline{Z}^N\right|\!\right\|_t - \left\|\!\left|\underline{Z}^N\right|\!\right\|_t}{\delta} = \frac{\left\|\!\left|\underline{Z}^N + \delta\underline{Y}^N\right|\!\right\| - \left\|\!\left|\underline{Z}^N\right|\!\right\|_t}{\delta} + \mathsf{Error}_\delta,$$

where $\mathsf{Error}_\delta \to 0$ as $\delta \to 0^+$. Subtracting $\partial_+ \left\|\!\left|\underline{Z}^N\right|\!\right\|_t \left(\underline{Y}^N\right)$ from both sides of the last equation and letting $\delta \to 0^+$ we get

$$\limsup_{\delta\to 0^+} \left(\frac{\left\|\!\left|T_{\delta u}\underline{Z}^N\right|\!\right\|_t - \left\|\!\left|\underline{Z}^N\right|\!\right\|_t}{\delta} - \partial_+ \left\|\!\left|\underline{Z}^N\right|\!\right\|_t \left(\underline{Y}^N\right) \right) \leq 0. \tag{4.19}$$

In a similar way we can prove that the reverse inequality of (4.19) is valid with the $\limsup$ replaced by a $\liminf$, which makes $\partial_+ \left\|\!\left|\underline{Z}^N\right|\!\right\|_t \left(\underline{Y}^N\right)$ the appropriate limit.

We now argue on how to pass the derivative inside the expectation in (4.18). The argument is similar to [Nor86]. For any function f we introduce the difference operator $\triangle_\delta f(\cdot) = f(\delta) - f(0)$.

We first show that the family of random variables

$$\frac{\triangle_\delta \left(\Phi\left(X^{N,\cdot}(t;x)\right)\chi\left(\left\|\!\left|T_{\cdot u}\underline{Z}^N\right|\!\right\|_t\right)\mathsf{A}^{\cdot}(t)\right)}{\delta} \quad \delta \in (0,1] \tag{4.20}$$

are uniformly integrable. We first write

$$\begin{aligned}
&\triangle_\delta \left(\Phi\left(X^{N,\cdot}(t;x)\right)\chi\left(\left\|\!\left|T_{\cdot u}\underline{Z}^N\right|\!\right\|_t\right)\mathsf{A}^{\cdot}(t)\right) \\
&\quad = \triangle_\delta \Phi\left(X^{N,\cdot}(t;x)\right)\chi(\left\|\!\left|T_{\delta u}\underline{Z}^N\right|\!\right\|_t)\mathsf{A}^{\delta}(t) \\
&\qquad + \Phi\left(X^{N,\delta}(t;x)\right)\triangle_\delta\, \chi\left(\left\|\!\left|T_{\cdot u}\underline{Z}^N\right|\!\right\|_t\right)\mathsf{A}^{\delta}(t) \\
&\qquad + \Phi\left(X^{N,\delta}(t;x)\right)\chi\left(\left\|\!\left|T_{\delta u}\underline{Z}^N\right|\!\right\|_t\right)\triangle_\delta\, \mathsf{A}^{\cdot}(t).
\end{aligned}$$

We treat each term on the right hand side separately. For the first term, we first use that $\Phi \in C^1_b(\mathcal{C}^{-\alpha_0})$ which prompts us to bound $\|X^{N,\delta}(t;x) - X^N(t;x)\|_{\mathcal{C}^{-\alpha_0}}$. By the mean value theorem we get

$$\|X^{N,\delta}(t;x) - X^N(t;x)\|_{\mathcal{C}^{-\alpha_0}} \leq \int_0^\delta \|\mathcal{D}X^{N,\lambda}(t;x)(w)\|_{\mathcal{C}^{-\alpha_0}}\,\mathrm{d}\lambda$$

where $\mathcal{D}X^{N,\lambda}(t;x)(w)$ solves (4.9) with $\underline{Z}^N$ replaced by $T_{\lambda u}\underline{Z}^N$. By (4.9) we get a bound on the quantity $\|\mathcal{D}X^{N,\lambda}(t;x)(w)\|_{\mathcal{C}^{-\alpha_0}}$ as soon as we have a bound on $\left|\!\left|\!\left|T_{\lambda u}\underline{Z}^N\right|\!\right|\!\right|$. The presence of the smooth indicator function yields a bound on $\left|\!\left|\!\left|T_{\delta u}\underline{Z}^N\right|\!\right|\!\right|$ which then by definition of the shift as well as the assumed boundedness of w yields a uniform bound on $\left|\!\left|\!\left|T_{\lambda u}\underline{Z}^N\right|\!\right|\!\right|$ for all $0 \le \lambda \le 1$. Hence we obtain a bound of the form

$$| \triangle_\delta \, \Phi\left(X^{N,\cdot}(t;x)\right) \chi\left(\left|\!\left|\!\left|T_{\delta u}\underline{Z}^N\right|\!\right|\!\right|_t\right) \mathsf{A}^\delta(t)| \le C\delta \|D\Phi\|_\infty \mathsf{A}^\delta(t),$$

where the constant C depends on w, χ and t. Arguing in the same way we get for the second term

$$|\Phi\left(X^{N,\delta}(t;x)\right) \triangle_\delta \, \chi\left(\left|\!\left|\!\left|T_{\cdot u}\underline{Z}^N\right|\!\right|\!\right|_t\right) \mathsf{A}^\delta(t)| \le C\delta \|\Phi\|_\infty \mathsf{A}^\delta(t).$$

Finally, for the third term we get using the mean value theorem for $\triangle_\delta \mathsf{A}^{\cdot}(t)$ that

$$|\Phi\left(X^{N,\delta}(t;x)\right) \chi\left(\left|\!\left|\!\left|T_{\delta u}\underline{Z}^N\right|\!\right|\!\right|_t\right) \triangle_\delta \, \mathsf{A}^{\cdot}(t)| \le C\|\Phi\|_\infty \int_0^\delta |\partial_\lambda \mathsf{A}^\lambda(t)| \, \mathrm{d}\lambda.$$

All the above imply that for every $p \ge 1$

$$\mathbb{E}\left|\frac{\triangle_\delta\left(\Phi\left(X^{N,\cdot}(t;x)\right) \chi\left(\left|\!\left|\!\left|T_{\cdot u}\underline{Z}^N\right|\!\right|\!\right|_t\right) \mathsf{A}^{\cdot}(t)\right)}{\delta}\right|^p \lesssim \sup_{\delta\in(0,1]} \mathbb{E}\mathsf{A}^\delta(t)^p + \sup_{\delta\in(0,1]} \mathbb{E}|\partial_\delta \mathsf{A}^\delta(t)|^p.$$

The key observation now is that $\mathsf{A}^\delta(t)^p = \mathsf{A}^{\delta p}(t) \exp\left\{\frac{p^2-p}{2}\int_0^t |\delta u(s)|^2 \, \mathrm{d}s\right\}$, where $\mathsf{A}^{\delta p}(t)$ is also an exponential martingale of expectation 1, while

$$\exp\left\{\frac{p^2-p}{2}\int_0^t |\delta u(s)|^2 \, \mathrm{d}s\right\}$$

is uniformly bounded in δ because of the almost sure bound on w. This implies that $\sup_{\delta\in(0,1]} \mathbb{E}\mathsf{A}^\delta(t)^p$ is bounded for any $p \ge 1$. Recalling the identity

$$\partial_\delta \mathsf{A}^\delta(t) = -\mathsf{A}^\delta(t)\left(\int_0^t u(s)\cdot \mathrm{d}\hat{W}_N(s) + \delta\int_0^t |u(s)|^2 \, \mathrm{d}s\right)$$

and using again the almost sure bound on w and the Cauchy–Schwarz inequality we have that

$$\mathbb{E}|\partial_\delta \mathsf{A}^\delta(t)|^p \lesssim \left(\mathbb{E}\mathsf{A}^\delta(t)^{2p}\right)^{\frac{1}{2p}} \left(\left(\mathbb{E}\left|\int_0^t u(s)\cdot \mathrm{d}\hat{W}_N(s)\right|^{2p}\right)^{\frac{1}{2p}} + 1\right).$$

The first term on the right hand side of the above inequality is uniformly bounded in δ as we discussed earlier while the second term can be bounded uniformly in δ using the Burkholder–Davis–Gundy inequality (see [RY99, Chapter 4, Theorem 4.1]) and the almost sure bound on w. Hence

$$\mathbb{E}\left|\frac{\triangle_\delta\left(\Phi\left(X^{N,\cdot}(t;x)\right)\chi\left(\left\|\left|T_{\cdot u}\underline{Z}^N\right|\right\|_t\right)\mathsf{A}^{\cdot}(t)\right)}{\delta}\right|^p<\infty,$$

uniformly in $\delta\in(0,1]$, for every $p\geq 1$, which implies uniform integrability of (4.20).

Using Vitali's convergence theorem (see [Bog07, Theorem 4.5.4]), we can now pass the derivative inside the expectation and differentiate by parts to obtain the identity

$$\begin{aligned}&\mathbb{E}\Big(D\Phi\left(X^{N,\delta}(t;x)\right)\left(\partial_\delta X^{N,\delta}(t;x)\right)\chi\left(\left\|\left|T_{\delta u}\underline{Z}^N\right|\right\|_t\right)\mathsf{A}^\delta(t)\Big)\Big|_{\delta=0}\\ &\quad=-\mathbb{E}\Big(\Phi\left(X^{N,\delta}(t;x)\right)\chi\left(\left\|\left|T_{\delta u}\underline{Z}^N\right|\right\|_t\right)\partial_\delta\mathsf{A}^\delta(t)\Big)\Big|_{\delta=0}\\ &\qquad-\mathbb{E}\Big(\Phi\left(X^{N,\delta}(t;x)\right)\partial_{\delta+}\chi\left(\left\|\left|T_{\delta u}\underline{Z}^N\right|\right\|_t\right)(w)\mathsf{A}^\delta(t)\Big)\Big|_{\delta=0}.\end{aligned}$$

The result follows since $\partial_\delta X^{N,\delta}(x;t)\Big|_{\delta=0}=\mathcal{D}X^N(t;x)(w)$ and $\partial_\delta\mathsf{A}^\delta(t)\Big|_{\delta=0}=-\int_0^t u(s)\cdot\mathrm{d}\hat{W}_N(s)$. □

Let $\{P_t^N:t\geq 0\}$ defined via the identity

$$P_t^N\Phi(x):=\mathbb{E}\Phi(X^N(t;x))\mathbf{1}_{\{t<T_*^N(x)\}}$$

for every $\Phi\in C_b(\mathcal{C}^{-\alpha_0})$, where we write $T_*^N(x)$ for the explosion time of v^N (see Proposition 4.4) dropping the dependence on $\mathfrak{l}^N$. We use (4.15) to prove the following proposition.

Proposition 4.11. *There exist $C>0$ and $\theta_1>0$ such that*

$$|P_t^N\Phi(x)-P_t^N\Phi(y)|\leq C\frac{1}{t^{\theta_1}}\|\Phi\|_\infty\|x-y\|_{\mathcal{C}^{-\alpha}}+2\|\Phi\|_\infty\mathbb{P}\left(t\geq\tau_{\frac{r}{2}}^N\right)\tag{4.21}$$

for every $x\in\mathcal{C}^{-\alpha_0}$, $y\in\bar{B}(x;1)$, $\Phi\in C_b^1(\mathcal{C}^{-\alpha_0})$ and $t\leq T_*\equiv T_*(R)$, where $T_*(R)$ is defined in Proposition 4.7 and $R=2\|x\|_{\mathcal{C}^{-\alpha_0}}+1$.

Proof. Let $\Phi\in C_b^1(\mathcal{C}^{-\alpha})$ and $t\leq T_*$. Then

$$\begin{aligned}&|P_t^N\Phi(x)-P_t^N\Phi(y)|\\ &\quad=\left|\mathbb{E}\left[\Phi\left(X^N(t;x)\right)\mathbf{1}_{\{t<T_*^N(x)\}}-\Phi\left(X^N(t;y)\right)\mathbf{1}_{\{t<T_*^N(y)\}}\right]\right|\end{aligned}$$

and the latter term is bounded by $I_1 + I_2$, where

$$I_1 := \left|\mathbb{E}\left[\Big(\Phi\left(X^N(t;x)\right) - \Phi\left(X^N(t;y)\right)\Big)\chi\left(|||\underline{Z}^N|||_t\right)\right]\right|$$
$$I_2 := \left|\mathbb{E}\left[\Big(\Phi\left(X^N(t;x)\right)\mathbf{1}_{\{t<T_*^N(x)\}} - \Phi\left(X^N(t;y)\right)\mathbf{1}_{\{t<T_*^N(y)\}}\Big)\chi^c\left(|||\underline{Z}^N|||_t\right)\right]\right|$$

for $\chi^c\left(|||\underline{Z}^N|||_t\right) = 1 - \chi\left(|||\underline{Z}^N|||_t\right)$. For the second term we have the bound $I_2 \leq 2\|\Phi\|_\infty \mathbb{P}\left(t \geq \tau^N_{\frac{r}{2}}\right)$ while by the mean value theorem we get that

$$\begin{aligned} I_1 &= \left|\mathbb{E}\left(\int_0^1 D\Phi\left(X^N(t;x+\lambda(y-x))\right)(y-x)\,\mathrm{d}\lambda\,\chi\left(|||\underline{Z}^N|||_t\right)\right)\right| \\ &= \left|\int_0^1 \mathbb{E}\Big(D\Phi\left(X^N(t;x+\lambda(y-x))\right)(y-x)\,\chi\left(|||\underline{Z}^N|||_t\right)\Big)\,\mathrm{d}\lambda\right|. \end{aligned}$$

For any $h_N \in \Pi_N L^2$ let w be such that $\partial_s w(s) = \left(\langle J^N_{0,s} h_N, e_m\rangle\right)_{|m|<N}$ for $s \leq \tau^N_r$ and 0 otherwise. Then $\partial_s w$ is an adapted process and by Proposition 4.7 there exists $C \equiv C(t) > 0$ such that $\|\partial_s w\|^2_{L^2\left([0,t];\mathbb{R}^{(2N-1)^2}\right)} \leq C$, $\mathbb{P}$-almost surely, for every initial condition $z_\lambda = x + \lambda(y-x)$ (recall that $J^N_{0,\cdot}$ depends on the initial condition and that $z_\lambda \in \bar{B}(x;1)$, for every $\lambda \in [0,1]$, thus the estimates in Proposition 4.7 hold uniformly in λ). Furthermore, $\mathcal{D}X^N(t;z_\lambda)(w) = tDX^N(t;z_\lambda)(h_N)$, for every $t \leq \tau^N_r$, and as in [Nor86] we can use (4.15) for this particular choice of w to obtain the following identity,

$$\begin{aligned} &\mathbb{E}\left(D\left[\Phi(X^N(t;z_\lambda))\right](h_N)\chi\left(|||\underline{Z}^N|||_t\right)\right) \\ &\quad = \frac{1}{t}\mathbb{E}\left(\Phi(X^N(t;z_\lambda))\int_0^t \langle J^N_{0,s}h_N,\, \mathrm{d}W_N(s)\rangle\chi\left(|||\underline{Z}^N|||_t\right)\right) \\ &\quad\quad - \frac{1}{t}\mathbb{E}\left(\Phi(X^N(t;z_\lambda))\partial_+\chi\left(|||\underline{Z}^N|||_t\right)(w)\right), \end{aligned} \tag{4.22}$$

where we slightly abuse the notation since, as we already mentioned, the operator $J^N_{0,\cdot}$ depends on the initial condition z_λ. In particular this is true for $h_N = \hat{\Pi}_N(y-x)$, hence

$$\begin{aligned} I_1 &\leq \frac{1}{t}\|\Phi\|_\infty \int_0^1 \mathbb{E}\left|\int_0^t \left\langle J^N_{0,s}\hat{\Pi}_N(y-x),\, \mathrm{d}W_N(s)\right\rangle \chi\left(|||\underline{Z}^N|||_t\right)\right| \mathrm{d}\lambda \\ &\quad + \frac{1}{t}\|\Phi\|_\infty \int_0^1 \mathbb{E}\left|\partial_+\chi\left(|||\underline{Z}^N|||_t\right)(w)\right| \mathrm{d}\lambda. \end{aligned}$$

Estimating the first term above we get

$$
\begin{aligned}
&\mathbb{E}\left|\int_0^t \left\langle J_{0,s}^N \hat{\Pi}_N(y-x),\, \mathrm{d}W_N(s)\right\rangle \chi\left(\left\|\!\left|\underline{Z}^N\right|\!\right\|_t\right)\right| \\
&\quad\le \mathbb{E}\left|\int_0^{t\wedge\tau_r^N} \left\langle J_{0,s}^N \hat{\Pi}_N(y-x),\, \mathrm{d}W_N(s)\right\rangle\right| \\
&\quad\le \left(\mathbb{E}\int_0^{t\wedge\tau_r^N} \|J_{0,s}^N \hat{\Pi}_N(y-x)\|_{L^2}^2 \,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\quad\le C t^{\frac{1}{2}-\gamma}\|x-y\|_{\mathcal{C}^{-\alpha_0}},
\end{aligned}
$$

where we use a Cauchy–Schwarz inequality and Itô's isometry in the second step and Proposition 4.7 in the third step. Here we use crucially, that the deterministic bound on $J_{0,s}^N$ provided in Proposition 4.7 holds uniformly in $N > 0$ (and in λ). Using the explicit form (4.16) of $\partial_+\chi\left(\left\|\!\left|\underline{Z}^N\right|\!\right\|_t\right)$ we also have the uniform in λ bound

$$
\mathbb{E}\left|\partial_+\chi\left(\left\|\!\left|\underline{Z}^N\right|\!\right\|_t\right)(w)\right| \le C t^{1-\gamma}\|x-y\|_{\mathcal{C}^{-\alpha_0}},
$$

since

$$
\begin{aligned}
&\partial_+\left\|\!\left|\underline{Z}^N\right|\!\right\|_t\left(Q_w(\cdot), 2\,\overset{\bullet}{|}{}^N Q_w(\cdot), \dots, n\,\overset{n-1}{\vee}{}^N Q_w(\cdot)\right) \\
&\quad\le C\left\|\!\left|\underline{Z}^N\right|\!\right\|_t t^{1-\gamma}\|x-y\|_{\mathcal{C}^{-\alpha_0}}
\end{aligned}
$$

and the fact that $\left\|\!\left|\underline{Z}^N\right|\!\right\|_t$ multiplied by $\partial_+\chi\left(\left\|\!\left|\underline{Z}^N\right|\!\right\|_t\right)$ is bounded by 1. Thus

$$
I_1 \le C\frac{1}{t^\gamma}\|\Phi\|_\infty\|x-y\|_{\mathcal{C}^{-\alpha_0}}
$$

and using both the bounds on I_1 and I_2 we get that for every $t \le T_*$

$$
|P_t^N\Phi(x) - P_t^N\Phi(y)| \le C\frac{1}{t^\gamma}\|\Phi\|_\infty\|x-y\|_{\mathcal{C}^{-\alpha_0}} + 2\|\Phi\|_\infty \mathbb{P}\left(t \ge \tau_{\frac{r}{2}}^N\right),
$$

which completes the proof. □

Given that the vector $\left(\overset{k}{\vee}{}^N\right)_{k=1}^n$ converges in law to $\left(\overset{k}{\vee}\right)_{k=1}^n$ on $C^{n,-\alpha}(0;T)$, for every $\alpha > 0$ and with respect to every norm $\left\|\!\left|\cdot\right|\!\right\|_{\alpha;\alpha';T}$, for every $T > 0$, we have that $\tau_{\frac{r}{2}}^N$ converges in law to $\tau_{\frac{r}{2}}$ when the mapping

$$
\underline{Z} \mapsto \inf\left\{t > 0 : \|\underline{Z}^{(1)}(t)\|_{\mathcal{C}^{-\alpha}} \vee \ldots \vee t^{(n-1)\alpha'}\|\underline{Z}^{(n)}(t)\|_{\mathcal{C}^{-\alpha}} > \frac{r}{2}\right\} \tag{4.23}
$$

is $\mathbb{P}$-almost surely continuous on the path $\left(\overset{k}{\vee}\right)_{k=1}^{n}$. But if

$$\mathsf{L} := \left\{ r \in (0,1] : \mathbb{P}\left(\text{(4.23) is discontinuous on } \left(\overset{k}{\vee}\right)_{k=1}^{n}\right) > 0 \right\}$$

and $M : [0,\infty) \to [0,\infty)$ is the mapping

$$t \mapsto \|\uparrow(t)\|_{\mathcal{C}^{-\alpha}} \vee \ldots \vee t^{(n-1)\alpha'} \|\overset{n}{\vee}(t)\|_{\mathcal{C}^{-\alpha}},$$

then

$$\mathsf{L} \subset \{ r \in (0,1] : \mathbb{P}\left(M \text{ has a local maximum at height } r\right) > 0 \}$$

and the last set is at most countable (see [MW17a, proof of Theorem 6.1]). Thus we can choose $r \in [\frac{1}{4}, 1]$ in (4.12) such that (4.23) is $\mathbb{P}$-almost surely continuous on $\left(\overset{k}{\vee}\right)_{k=1}^{n}$. This implies convergence in law of $\tau^N_{\frac{r}{2}}$ to $\tau_{\frac{r}{2}}$, thus

$$\limsup_{N\to\infty} \mathbb{P}\left(t \geq \tau^N_{\frac{r}{2}}\right) \leq \mathbb{P}\left(t \geq \tau_{\frac{r}{2}}\right).$$

Notice that global existence of $v(t)$ (see Theorem 3.12) implies global existence of $X(t;x)$ and in particular existence for every $t \leq T_*(R)$. Using Propositions 2.3 and 3.7, $\liminf_{N\to\infty} T^N_* \geq T_*(R)$ and $\sup_{t \leq T^N_* \wedge T_*(R)} \|X^N(t;x^N) - X(t;x)\|_{\mathcal{C}^{-\alpha_0}} \to 0$ $\mathbb{P}$-almost surely, for every $x \in \mathcal{C}^{-\alpha}$. By the dominated convergence theorem $P^N_t \Phi(x)$ converges to $P_t\Phi(x)$, for every $\Phi \in C^1_b(\mathcal{C}^{-\alpha_0})$, and we retrieve (4.21) for the limiting semigroup P_t, for every $t \leq T_*(R)$, in the form

$$|P_t\Phi(x) - P_t\Phi(y)| \leq C\frac{1}{t^{\theta_1}}\|\Phi\|_\infty \|x-y\|_{\mathcal{C}^{-\alpha_0}} + 2\|\Phi\|_\infty \mathbb{P}\left(t \geq \tau_{\frac{r}{2}}\right). \tag{4.24}$$

Remark 4.12. The above argument can be modified to retrieve (4.24) without using global existence for the limiting process. In this case, one can define the semigroup P_t by introducing a cemetery state for the process $X(t;x)$.

We finally prove the following theorem. Below we denote by $\|\mu_1 - \mu_2\|_{\mathrm{TV}}$ the total variation distance of two probability measures $\mu_1, \mu_2 \in \mathcal{M}_1(\mathcal{C}^{-\alpha_0})$ given by

$$\|\mu_1 - \mu_2\|_{\mathrm{TV}} := \frac{1}{2} \sup_{\|\Phi\|_\infty \leq 1} |\mathbb{E}_{\mu_1}\Phi - \mathbb{E}_{\mu_2}\Phi|.$$

Theorem 4.13. *There exists $\theta \in (0,1)$ and $\sigma > 0$ such that for every $x \in \mathcal{C}^{-\alpha_0}$ and $y \in \bar{B}(x;1)$*

$$\|P^*_t\delta_x - P^*_t\delta_y\|_{\mathrm{TV}} \leq C(1 + \|x\|_{\mathcal{C}^{-\alpha_0}})^\sigma \|x-y\|^\theta_{\mathcal{C}^{-\alpha_0}},$$

for every $t \geq 1$. In particular, for every $t \geq 1$, P_t is locally uniformly θ-Hölder continuous with respect to the total variation distance in $\mathcal{C}^{-\alpha_0}$.

Proof. Let $R = 2\|x\|_{\mathcal{C}^{-\alpha_0}} + 1$. By an approximation argument (see [DPZ96, Lemma 7.1.5]) (4.24) is equivalent to

$$\|P_t^*\delta_x - P_t^*\delta_y\|_{\mathrm{TV}} \leq \frac{C}{2}\frac{1}{t^{\theta_1}}\|x-y\|_{\mathcal{C}^{-\alpha_0}} + \mathbb{P}\left(t \geq \tau_{\frac{r}{2}}\right),$$

for every $t \leq T_*$ and $y \in \bar{B}(x;1)$. Notice that

$$\mathbb{P}\left(t \geq \tau_{\frac{r}{2}}\right) \leq \mathbb{P}\left(\|\!|\underline{Z}\|\!|_{\alpha;\alpha';t} > \frac{r}{2}\right)$$

and by Theorem 2.1

$$\mathbb{P}\left(\|\!|\underline{Z}\|\!|_{\alpha;\alpha';t} > r\right) \leq C\frac{1}{r}t^{\theta_2},$$

for some $\theta_2 \in (0,1)$. Since we can assume that $T_* \leq 1$, we have that

$$\|P_1^*\delta_x - P_1^*\delta_y\|_{\mathrm{TV}} \leq \|P_{T_*}^*\delta_x - P_{T_*}^*\delta_y\|_{\mathrm{TV}}$$

where

$$\|P_{T_*}^*\delta_x - P_{T_*}^*\delta_y\|_{\mathrm{TV}} \leq \inf_{t \leq T_*}\left\{C_1\frac{1}{t^{\theta_1}}\|x-y\|_{\mathcal{C}^{-\alpha_0}} + C_2\frac{1}{r}t^{\theta_2}\right\}.$$

Let $f(t) := C_1\frac{1}{t^{\theta_1}}\|x-y\|_{\mathcal{C}^{-\alpha_0}} + C_2\frac{1}{r}t^{\theta_2}$, $t > 0$, and notice that for

$$t_0 = \left(\frac{\theta_1 C_1 r\|x-y\|_{\mathcal{C}^{-\alpha_0}}}{\theta_2 C_2}\right)^{\frac{1}{\theta_1+\theta_2}},$$

$f(t_0) = \inf_{t>0} f(t)$. If $t_0 \leq T_*$, then there exists $C \equiv C(\theta_1, \theta_2, r)$ such that

$$\|P_{T_*}^*\delta_x - P_{T_*}^*\delta_y\|_{\mathrm{TV}} \leq f(t_0) = C\|x-y\|_{\mathcal{C}^{-\alpha_0}}^{\frac{\theta_2}{\theta_1+\theta_2}}.$$

Otherwise $t_0 \geq T_*$ and using

$$\begin{aligned}
\|P_{T_*}^*\delta_x - P_{T_*}^*\delta_y\|_{\mathrm{TV}} &\leq C_1\frac{1}{(T_*)^{\theta_1}}\|x-y\|_{\mathcal{C}^{-\alpha_0}} + C_2\frac{1}{r}(T_*)^{\theta_2} \\
&\leq C_1\frac{1}{(T_*)^{\theta_1}}\|x-y\|_{\mathcal{C}^{-\alpha_0}} + C_2\frac{1}{r}t_0^{\theta_2} \\
&= C_1\frac{1}{(T_*)^{\theta_1}}\|x-y\|_{\mathcal{C}^{-\alpha_0}} + \tilde{C}_2\frac{1}{r}\|x-y\|_{\mathcal{C}^{-\alpha_0}}^{\frac{\theta_2}{\theta_1+\theta_2}}
\end{aligned}$$

and the explicit estimate of T_* (see (3.10)) we get

$$\begin{aligned}
\|P_{T_*}^*\delta_x - P_{T_*}^*\delta_y\|_{\mathrm{TV}} &\leq \tilde{C}_1(1+R)^{3\frac{\theta_1}{\theta}}\|x-y\|_{\mathcal{C}^{-\alpha_0}} + \tilde{C}_2\frac{1}{r}\|x-y\|_{\mathcal{C}^{-\alpha_0}}^{\frac{\theta_2}{\theta_1+\theta_2}} \\
&\leq C(1+R)^{3\frac{\theta_1}{\theta}+\frac{\theta_1}{\theta_1+\theta_2}}\|x-y\|_{\mathcal{C}^{-\alpha_0}}^{\frac{\theta_2}{\theta_1+\theta_2}}
\end{aligned}$$

for a constant $C \equiv C(\theta_1, \theta_2, r)$ and some $\theta > 0$ as in (3.10). Combining all the above we finally get

$$\|P_1^*\delta_x - P_1^*\delta_y\|_{\mathrm{TV}} \leq C(1+R)^{3\frac{\theta_1}{\theta}+\frac{\theta_1}{\theta_1+\theta_2}} \|x-y\|_{\mathcal{C}^{-\alpha_0}}^{\frac{\theta_2}{\theta_1+\theta_2}},$$

which completes the proof. □

4.4 A Support Theorem

From now on we restrict ourselves in the case $n = 3$ (see Remark 4.15). In this section following [CF16] we prove a support theorem for the solutions to (3.1) for $n = 3$. For simplicity, we also assume that the torus $\mathbb{T}^2$ has size 1, that is, $\mathbb{T}^2 = \mathbb{R}^2/L\mathbb{Z}^2$ for $L = 1$. However, all the proofs in this section can be carried out for general L, by possibly making the explicit constants to depend on L.

We consider $\underline{Y} = \left(\vee^{k}_{-\infty}\right)_{k=1}^{3}$ (see (2.5) for the definition of $\vee^{k}_{-\infty}$) as an element of $C([0,T];\mathcal{C}^{-\alpha})^3$ endowed with the norm $|||\cdot|||_{\alpha;0;T}$, for some $\alpha \in (0,1)$, given by

$$|||\underline{Y}|||_{\alpha;0;T} := \max_{k=1,2,3} \left\{ \sup_{t\leq T} \|\vee^{k}_{-\infty}(t)\|_{\mathcal{C}^{-\alpha}} \right\}.$$

Here we are allowed to use a non-weighted norm since there is no blow-up of $\vee^{k}_{-\infty}$ at zero (see Theorem 2.1). We furthermore let

$$\mathscr{H}(T) := \left\{ h\big|_{[0,T]} : h(t) = \int_{-\infty}^{t} S_1(t-r) f(r)\,\mathrm{d}r,\ t \geq 0 \text{ and } f \in L^2(\mathbb{R}\times\mathbb{T}^2) \right\}.$$

It is worth mentioning that $\mathscr{H}(T)$ consists of those L^2-integrable space-time functions with zero initial datum and with one derivative in time and two derivatives in space in L^2.

Lemma 4.14. *Let $\{C_m\}_{m\geq 1}$ be a sequence of positive numbers such that $C_m \leq C(m+1)$. Then there exists a sequence of smooth functions $\{f_m\}_{m\geq 1}$ such that*

i. $f_m \in \mathcal{C}^{-\alpha}$, for every $\alpha \in (0,1)$.

ii. $|\langle f_m, e_l\rangle|^2 = C_m$ if $l = 2^m(1,1)$ or $l = -2^m(1,1)$ and 0 otherwise.

iii. For every $n = 1,2,3$, $\mathcal{H}_n(f_m, C_m) \to 0$ in $\mathcal{C}^{-\alpha}$, for every $\alpha \in (0,1)$.

Proof. Let

$$f_m(z) := \frac{\mathrm{e}^{2\pi \mathrm{i} 2^m z_0 \cdot z} + \mathrm{e}^{-2\pi \mathrm{i} 2^m z_0 \cdot z}}{2^{1/2}} C_m^{1/2},$$

where $z_0 = (1,1) \in \mathbb{Z}^2$, $z \in \mathbb{T}^2$. Then for $\kappa \geq -1$

$$\begin{aligned}
\delta_\kappa f_m(z) &= \frac{C_m^{1/2}}{2^{1/2}} \mathbf{1}_{\{m=\kappa\}} \left(\mathrm{e}^{2\pi \mathrm{i} 2^m z_0 \cdot z} + \mathrm{e}^{-2\pi \mathrm{i} 2^m z_0 \cdot z}\right) \\
\delta_\kappa f_m(z)^2 - C_m &= \frac{C_m}{2} \mathbf{1}_{\{m+1=\kappa\}} \left(\mathrm{e}^{2\pi \mathrm{i} 2^{m+1} z_0 \cdot z} + \mathrm{e}^{-2\pi \mathrm{i} 2^{m+1} z_0 \cdot z}\right) \\
\delta_\kappa f_m(z)^3 &= \frac{C_m^{3/2}}{2^{3/2}} \Big[\chi_\kappa(2^m 3 z_0) \left(\mathrm{e}^{2\pi \mathrm{i} 2^m 3 z_0 \cdot z} + \mathrm{e}^{-2\pi \mathrm{i} 2^m 3 z_0 \cdot z}\right) \\
&\qquad + \mathbf{1}_{\{m=\kappa\}} 3 \left(\mathrm{e}^{2\pi \mathrm{i} 2^m z_0 \cdot z} + \mathrm{e}^{-2\pi \mathrm{i} 2^m z_0 \cdot z}\right)\Big].
\end{aligned}$$

Notice here we have used the convenient fact that the particular choice of z_0 has the property that $\chi_\kappa(2^m z_0) = \mathbf{1}_{\{m=\kappa\}}$. Thus we have

$$\begin{aligned}
&\|f_m\|_{\mathcal{C}^{-\alpha}} \lesssim C_m^{1/2} 2^{-\alpha m} \\
&\|f_m^2 - C_m\|_{\mathcal{C}^{-\alpha}} \lesssim C_m 2^{-\alpha m} \\
&\|f_m^3 - 3C_m f_m\|_{\mathcal{C}^{-\alpha}} \lesssim C_m^{3/2} 2^{-\alpha m}.
\end{aligned}$$

Given that $C_m \lesssim m+1$ all the above quantities tend to 0 as $m \to \infty$, which completes the proof. $\square$

Remark 4.15. The sequence $\{f_m\}_{m\geq 1}$ introduced in the lemma above satisfies property iii for every odd n. For such n every term appearing in $\mathcal{H}_n(f_m, C_m)$ is a multiple of $C_m^{k_1} e_{2^m k_2 z_0}$ for a $k_2 \neq 0$ and the fast (exponential) decay of $\|e_{2^m k_2 z_0}\|_{\mathcal{C}^{-\alpha}}$ compensates the slow (polynomial) growth of $C_m^{k_1}$. However, for even n this property fails, because for such n the quantity $\mathcal{H}_n(f_m, C_m)$ contains a multiple of C_m^n which does not need to vanish. We suspect, that a first step in order to generalise Theorem 4.16 to the case of general n would be the construction of a sequence $\{f_m\}_{m\geq 1}$ with Fourier support on an annulus and such that

$$\int_{\mathbb{T}^2} f_m(z)^k \,\mathrm{d}z = \mathcal{H}_k(0, C_m),$$

for every $k \geq 1$.

We now prove the following support theorem.

Theorem 4.16. *Let $\mathbb{P}_{\underline{Y}}$ be the law of $\underline{Y}$ in $C([0,T];\mathcal{C}^{-\alpha})^3$ endowed with the norm $|\!|\!|\cdot|\!|\!|_{\alpha;0;T}$. Then*

$$\operatorname{supp}\mathbb{P}_{\underline{Y}} = \overline{\left\{ \left(\mathcal{H}_k(h,\Re)\right)_{k=1}^3 : h \in \mathscr{H}(T),\ \Re \geq 0 \right\}}^{|\!|\!|\cdot|\!|\!|_{\alpha;0;T}}.$$

Proof. For $h \in \mathscr{H}(T)$ and $\underline{Y} \in C^{3,-\alpha}(0;T)$ let T_h be the shift

$$T_h Y^{(k)} = \sum_{j=0}^{k} \binom{k}{j} h^j \, Y^{(k-j)}, \quad k = 1,2,3,$$

where we use again the convention that $Y^{(0)} \equiv 1$, and write $T_h\underline{Y} = \left(T_h Y^{(k)}\right)_{k=1}^3$. Here we slightly abuse the notation since the action of T_h on $Y^{(k)}$ needs information on the lower order terms.

As in [CF16], it suffices to prove that $(0,-\Re,0) \in \operatorname{supp}\mathbb{P}_{\underline{Y}}$, for every $\Re \geq 0$. Then, given that shifts of the initial probability measure in the direction of the Cameron–Martin space generate equivalent probability measures, for every $h \in \mathscr{H}(T)$, $T_h(0,-\Re,0) \in \operatorname{supp}\mathbb{P}_{\underline{Y}}$, which completes the proof since by the definition of T_h the latter is equal to $(\mathcal{H}_k(h,\Re))_{k=1}^3$ (see also [CF16, Corollary 3.10]).

For $\lambda > 0$ and $\rho_{\lambda 2^m}(z) = \sum_{|\bar{m}|<\lambda 2^m} e_{\bar{m}}(z)$ we let

$$\Upsilon^m_{-\infty}(t,z) := \langle \Upsilon_{-\infty}(t), \rho_{\lambda 2^m}(z-\cdot)\rangle, \quad \Re_m := \mathbb{E}\Upsilon^m_{-\infty}(t,0)^2,$$

where $\Upsilon^m_{-\infty}(t)$ coincides with $\Upsilon^N_{-\infty}(t)$ in Section 2.2 for $N = \lambda 2^m$. Notice that for $\Re \geq 0$ there exists $m_0 \equiv m_0(\Re) > 1$ such that $\Re_m - \Re > 0$, for every $m \geq m_0$ (recall that $\Re_m \sim \log m$). Thus if we set $C_m = 0$ for $m \leq m_0$ and $C_m = \Re_m - \Re$ otherwise, then $C_m \geq 0$ and $C_m \lesssim m+1$. We consider f_m as in Lemma 4.14 for this particular choice of C_m and for $\lambda_m = 1 + 4\pi^2 2^{2m}|z_0|^2$ we let

$$h_m(t) := \left(1 - \mathrm{e}^{-\lambda_m(t+1)}\right) f_m,$$

for $t \in [0,T]$. Then $h_m \in \mathscr{H}(T)$ since $h_m(t) = \frac{1}{\lambda_m}\int_{-1}^t S_1(t-r) f_m \,\mathrm{d}r$ and we furthermore have the uniform in t estimates

$$\begin{aligned}
&\|h_m(t)\|_{\mathcal{C}^{-\alpha}} \leq \|f_m\|_{\mathcal{C}^{-\alpha}} \\
&\|h_m(t)^2 - C_m\|_{\mathcal{C}^{-\alpha}} \leq \|f_m^2 - C_m\|^2_{\mathcal{C}^{-\alpha}} + 2\mathrm{e}^{-\lambda_m} C_m \\
&\|h_m(t)^3\|_{\mathcal{C}^{-\alpha}} \leq \|f_m^3\|_{\mathcal{C}^{-\alpha}}.
\end{aligned}$$

Finally, we define

$$w_m := -\mathbf{1}^m_{-\infty} - h_m.$$

We prove that the following convergences hold in every stochastic L^p space of random variables taking values in $C([0,T];\mathcal{C}^{-\alpha})$,

$$T_{w_m}\mathbf{1}_{-\infty} \to 0,\ T_{w_m}\mathbf{V}_{-\infty} \to -\mathfrak{R},\ T_{w_m}\mathbf{W}_{-\infty} \to 0.$$

By the same argument as in [CF16, Lemma 3.13] this implies the result. For the reader's convenience, we sketch the argument here. Since $w_m \in \mathscr{H}(T)$, by Lemma [CF16, Corollary 3.10] there exists a subset Ω' of Ω of probability one such that for every $\omega \in \Omega'$

$$(T_{w_m(\omega)}\mathbf{1}_{-\infty}(\omega), T_{w_m(\omega)}\mathbf{V}_{-\infty}(\omega), T_{w_m(\omega)}\mathbf{W}_{-\infty}(\omega)) \in \operatorname{supp}\mathbb{P}_{\underline{Y}}$$

for every $m \geq 1$. Given that $\operatorname{supp}\mathbb{P}_{\underline{Y}}$ is closed under the norm $|||\cdot|||_{\alpha;0;T}$, we can conclude that $(0,-\mathfrak{R},0) \in \operatorname{supp}\mathbb{P}_{\underline{Y}}$ as soon as the above convergence holds for a single element $\omega \in \Omega'$. The stochastic L^p convergence implies almost sure convergence along a subsequence which is sufficient.

The convergence of $T_{w_m}\mathbf{1}_{-\infty}$ to 0 is an immediate from Proposition 2.3 and Lemma 4.14.

If we compute the corresponding shift for $\mathbf{V}_{-\infty}$ we get

$$\begin{aligned} T_{w_m}\mathbf{V}_{-\infty}(t) = \mathbf{V}_{-\infty}(t) &+ \left((\mathbf{1}^m_{-\infty}(t))^2 - \mathfrak{R}_m\right) - 2\left(\mathbf{1}_{-\infty}(t)\mathbf{1}^m_{-\infty}(t) - \mathfrak{R}_m\right) \\ &+ 2\mathbf{1}^m_{-\infty}(t)h_m(t) + \mathcal{H}_2(h_m(t),\mathfrak{R}_m), \end{aligned}$$

where we also add and subtract $2\mathfrak{R}_m$ where necessary. If we choose λ sufficiently small we can ensure that

$$\mathbf{1}^m_{-\infty}(t) \circ h_m(t) \equiv 0,$$

where $\mathbf{1}^m_{-\infty,t} \circ h_m(t)$ is the resonant term define in (A.9). Using the Bony estimates (see Proposition A.6), Lemma 4.14 and the fact that $\mathbf{1}^m_{-\infty}$ is bounded in every stochastic L^p space taking values in $C([0,T];\mathcal{C}^{-\alpha})$ we get that $\mathbf{1}^m_{-\infty}(t)h_m(t) \to 0$. For the term

$$\mathbf{V}_{-\infty}(t) + \left(\mathbf{1}^m_{-\infty}(t)^2 - \mathfrak{R}_m\right) - 2\left(\mathbf{1}_{-\infty}(t)\mathbf{1}^m_{-\infty}(t) - \mathfrak{R}_m\right)$$

by Proposition 2.3 it suffices to compute the limit of $\uparrow_{-\infty}(t)\uparrow^{m}_{-\infty}(t) - \Re_m$. We only give a sketch of the proof since the idea is similar to the one in the proof of Proposition 2.3 (see Appendix E). Notice that for $m' > m$, $\mathbb{E}\uparrow^{m'}_{-\infty}(t)\uparrow^{m}_{-\infty}(t) = \Re_m$, thus using [Nua06, Proposition 1.1.2] we have that

$$\uparrow^{m'}_{-\infty}(t)\uparrow^{m}_{-\infty}(t) - \Re_m = \uparrow^{m'}_{-\infty}(t) \otimes \uparrow^{m}_{-\infty}(t),$$

where $\otimes$ denotes the renormalised product given by

$$\begin{aligned}
&(\vee^{j})^{m'}_{-\infty} \otimes (\vee^{i})^{m}_{-\infty}(t,z) \\
&\quad := 2^{\frac{i+j}{2}} \int_{\{(-\infty,t]\times\mathbb{T}^2\}^{j+i}} \prod_{\substack{1\le j'\le j \\ 1\le i'\le i}} H_{m'}(t-r_{i'}, z-z_{i'}) H_m(t-r_{j'}, z-z_{j'}) \\
&\qquad \times \xi(\otimes_{k=1}^{i+j} \mathrm{d}z_k, \otimes_{k=1}^{i+j} \mathrm{d}r_k),
\end{aligned}$$

for every $z \in \mathbb{T}^2$ and $i, j \ge 1$. In the same spirit as in the proof of Proposition 2.3 (see Appendix E) we can prove that

$$\lim_{m\to\infty} \lim_{m'\to\infty} \mathbb{E} \sup_{t\le T} \|\uparrow^{m'}_{-\infty}(t) \otimes \uparrow^{m}_{-\infty}(t) - \vee_{-\infty}(t)\|^p_{\mathcal{C}^{-\alpha}} = 0,$$

for every $p \ge 2$. Combining the above with the fact that $\sup_{t\le T} \|h_m(t)^2 - (\Re_m - \Re)\|_{\mathcal{C}^{-\alpha}}$ converges to 0, we obtain that $T_{w_m} \vee_{-\infty} \to -\Re$.

For the term $T_{w_m} \Psi_{-\infty}(t)$, by adding and subtracting multiples of $\Re_m \uparrow^{m}_{-\infty,t}$ and $\Re_m$ where necessary we have that

$$\begin{aligned}
T_{w_m} \Psi_{-\infty}(t) = {} & \Psi_{-\infty}(t) - \left(\uparrow^{m}_{-\infty}(t)^3 - 3\Re_m \uparrow^{m}_{-\infty}(t)\right) \\
& - 3\left(\uparrow^{m}_{-\infty}(t)\vee_{-\infty}(t) - 2\Re_m \uparrow^{m}_{-\infty}(t)\right) \\
& + 3\left(\uparrow_{-\infty}(t)\uparrow^{m}_{-\infty}(t)^2 - 3\Re_m \uparrow^{m}_{-\infty}(t)\right) \\
& + 3h_m(t)\left(\vee_{-\infty}(t) + (\uparrow^{m}_{-\infty}(t)^2 - \Re_m) - 2(\uparrow_{-\infty}(t)\uparrow^{m}_{-\infty}(t) - \Re_m)\right) \\
& + 3h_m(t)^2\left(\uparrow_{-\infty}(t) - \uparrow^{m}_{-\infty}(t)\right) + \mathcal{H}_3(h_m(t), \Re_m).
\end{aligned}$$

For the terms $\uparrow^{m}_{-\infty}(t)\vee_{-\infty}(t) - 2\Re_m \uparrow^{m}_{-\infty}(t)$, $\uparrow_{-\infty}(t)\uparrow^{m}_{-\infty}(t)^2 - 3\Re_m \uparrow^{m}_{-\infty}(t)$ using again [Nua06, Proposition 1.1.2] for $m' > m$ we have that

$$\begin{aligned}
&\uparrow^{m}_{-\infty}(t)\vee^{m'}_{-\infty}(t) - 2\Re_m \uparrow^{m}_{-\infty}(t) = \uparrow^{m}_{-\infty}(t) \otimes \vee^{m'}_{-\infty}(t) + 2\Re_m(\uparrow^{m'}_{-\infty}(t) - \uparrow^{m}_{-\infty}(t)) \\
&\uparrow^{m'}_{-\infty}(t)\uparrow^{m}_{-\infty}(t)^2 - 3\Re_m \uparrow^{m}_{-\infty}(t) = \uparrow^{m'}_{-\infty}(t) \otimes \vee^{m}_{-\infty}(t) + \Re_m(\uparrow^{m'}_{-\infty}(t) - \uparrow^{m}_{-\infty}(t)).
\end{aligned}$$

If we proceed again in the spirit of the proof of Proposition 2.3 (see Appendix E) we obtain that

$$
\begin{aligned}
&\lim_{m\to\infty}\lim_{m'\to\infty}\mathbb{E}\sup_{t\le T}\|\mathbf{I}^{m}_{-\infty}(t)\otimes\mathbf{V}^{m'}_{-\infty}(t)-\mathbf{\Psi}_{-\infty}(t)\|^p_{\mathcal{C}^{-\alpha}}=0\\
&\lim_{m\to\infty}\lim_{m'\to\infty}\mathbb{E}\sup_{t\le T}\|\mathbf{I}^{m'}_{-\infty}(t)\otimes\mathbf{V}^{m}_{-\infty}(t)-\mathbf{\Psi}_{-\infty}(t)\|^p_{\mathcal{C}^{-\alpha}}=0\\
&\lim_{m\to\infty}\lim_{m'\to\infty}(\mathfrak{R}_m)^p\,\mathbb{E}\sup_{t\le T}\|\mathbf{I}^{m'}_{-\infty}(t)-\mathbf{I}^{m}_{-\infty}(t)\|^p_{\mathcal{C}^{-\alpha}}=0,
\end{aligned}
$$

for every $p\geq 2$. It remains to handle the terms

$$
h_m(t)\Big(\mathbf{V}_{-\infty}(t)-(\mathbf{I}_{-\infty}(t)\mathbf{I}^{m}_{-\infty}(t)-\mathfrak{R}_m)\Big), \tag{4.25}
$$

$$
h_m(t)\Big(\mathbf{I}^{m}_{-\infty}(t)^2-\mathfrak{R}_m-(\mathbf{I}_{-\infty}(t)\mathbf{I}^{m}_{-\infty}(t)-\mathfrak{R}_m)\Big) \tag{4.26}
$$

and

$$
h_m(t)^2(\mathbf{I}_{-\infty}(t)-\mathbf{I}^{m}_{-\infty}(t)). \tag{4.27}
$$

We only show that (4.25) converges to 0 since (4.26) and (4.27) can be handled in a similar way. In particular due to Bony estimates (see Proposition A.6) it suffices to prove that the resonant term

$$
\begin{aligned}
&h_m(t)\circ\big(\mathbf{V}_{-\infty}(t)-(\mathbf{I}_{-\infty}(t)\mathbf{I}^{m}_{-\infty}(t)-\mathfrak{R}_m)\big)\\
&\quad=\sum_{|\kappa_1-\kappa_2|\le 1}\delta_{\kappa_1}h_m(t)\delta_{\kappa_2}\left[\mathbf{V}_{-\infty}(t)-(\mathbf{I}_{-\infty}(t)\mathbf{I}^{m}_{-\infty}(t)-\mathfrak{R}_m)\right]
\end{aligned}
$$

converges to 0. Since the Fourier modes of h_m are localised at the points $2^m z_0$ and $-2^m z_0$ we have that

$$
\begin{aligned}
&h_m(t)\circ\big(\mathbf{V}_{-\infty}(t)-(\mathbf{I}_{-\infty}(t)\mathbf{I}^{m}_{-\infty}(t)-\mathfrak{R}_m)\big)\\
&\quad=h_m(t)\sum_{i=-1,0,1}\delta_{m+i}\left[\mathbf{V}_{-\infty}(t)-(\mathbf{I}_{-\infty}(t)\mathbf{I}^{m}_{-\infty}(t)-\mathfrak{R}_m)\right].
\end{aligned}
$$

Let $\kappa\geq -1$ and $Y_m(t)=\mathbf{V}_{-\infty}(t)-(\mathbf{I}_{-\infty}(t)\mathbf{I}^{m}_{-\infty}(t)-\mathfrak{R}_m)$. Then, for $i=-1,0,1$,

$$
\begin{aligned}
&\mathbb{E}\delta_\kappa[h_m(t_1)\delta_{m+i}Y_m(t_1)](z_1)\delta_\kappa[h_m(t_2)\delta_{m+i}Y_m(t_2)](z_2)\\
&\quad=\int_{\mathbb{T}^2\times\mathbb{T}^2}C_{m,i}(t_1-t_2,\bar z_1-\bar z_2)\eta_\kappa(z_1-\bar z_1)\eta_\kappa(z_2-\bar z_2)h_m(t_1,\bar z_1)h_m(t_2,\bar z_2)\\
&\qquad\times\,\mathrm{d}\bar z_1\,\mathrm{d}\bar z_2,
\end{aligned}
$$

where

$$C_{m,i}(t_1 - t_2, \bar{z}_1 - \bar{z}_2) = \mathbb{E}\delta_{m+i}[Y_m(t_1)](\bar{z}_1)\delta_{m+i}[Y_m(t_2)](\bar{z}_2).$$

For $m' > m$ using [Nua06, Proposition 1.1.2] we have that $\Psi^{m'}_{-\infty}(t)\Psi^{m}_{-\infty}(t) - \mathfrak{R}_m = \Psi^{m'}_{-\infty}(t) \otimes \Psi^{m}_{-\infty}(t)$. Let $Y_{m,m'}(t) = \mathbf{V}_{-\infty}(t) - \Psi^{m'}_{-\infty}(t) \otimes \Psi^{m}_{-\infty}(t)$ and notice that

$$\begin{aligned}
&\mathbb{E}\delta_{m+i}[Y_{m,m'}(t_1)](\bar{z}_1)\delta_{m+i}[Y_{m,m'}(t_2)](\bar{z}_2) \\
&\quad = C \sum_{\substack{|l_1|>\lambda 2^{m'} \\ |l_2|>\lambda 2^m}} \prod_{j=1,2} \frac{1-\mathrm{e}^{-I_{l_j}|t_2-t_1|}}{2I_{l_j}} |\chi_{m+i}(l_1+l_2)|^2 e_{l_1+l_2}(\bar{z}_1 - \bar{z}_2),
\end{aligned}$$

for some constant C independent of m and m' and $I_{l_j} = 1 + 4\pi^2|l_j|^2$. Then for every $\gamma \in (0, \frac{1}{2})$ by a change of variables

$$\begin{aligned}
&\int_{\mathbb{T}^2\times\mathbb{T}^2} C_{m,m',i}(t_1 - t_2, \bar{z}_1 - \bar{z}_2)\eta_\kappa(z_1 - \bar{z}_1)\eta_\kappa(z_2 - \bar{z}_2)h_m(t_1, \bar{z}_1)h_m(t_2, \bar{z}_2) \\
&\times \, \mathrm{d}\bar{z}_1 \, \mathrm{d}\bar{z}_2 \\
&\quad \lesssim \underbrace{\left(\sum_{\substack{l\in\mathcal{A}_{2^{m+i}} \\ l+2^m z_0 \in \mathcal{A}_{2^\kappa}}} K^\gamma \star^2_{>\lambda 2^m} K^\gamma(l) + \sum_{\substack{l\in\mathcal{A}_{2^{m+i}} \\ l-2^m z_0 \in \mathcal{A}_{2^\kappa}}} K^\gamma \star^2_{>\lambda 2^m} K^\gamma(l) \right)}_{I} \\
&\qquad \times (m+1)|t_1 - t_2|^{2\gamma},
\end{aligned}$$

where $K^\gamma(l) = \frac{1}{(1+|l|^2)^{1-\gamma}}$ and $C_{m,m',i}$ is defined as $C_{m,i}$ with Y_m replaced by $Y_{m,m'}$. By Corollary C.3

$$I \lesssim \sum_{\substack{l\in\mathcal{A}_{2^{m+i}} \\ l+2^m z_0 \in \mathcal{A}_{2^\kappa}}} \frac{1}{(1+|l|^2)^{1-2\gamma}} + \sum_{\substack{l\in\mathcal{A}_{2^{m+i}} \\ l-2^m z_0 \in \mathcal{A}_{2^\kappa}}} \frac{1}{(1+|l|^2)^{1-2\gamma}},$$

thus for every $\varepsilon > 2\gamma$

$$I \lesssim 2^{2\varepsilon k} \sum_{l\in\mathbb{Z}^2} \frac{1}{(1+|l|^2)^{1-2\gamma}} \frac{1}{(1+|l+2^m z_0|^2)^{\varepsilon}}.$$

Using Corollary C.3 we obtain

$$\begin{aligned}
&\mathbb{E}\delta_\kappa[h_m(t_1)\delta_{m+i}Y_{m,m'}(t_1)](z_1)\delta_\kappa[h_m(t_2)\delta_{m+i}Y_{m,m'}(t_2)](z_2) \\
&\quad \lesssim \frac{2^{2\varepsilon\kappa}(m+1)}{(1+|2^m z_0|^2)^{\varepsilon-2\gamma}}|t_1 - t_2|^{2\gamma},
\end{aligned}$$

for every $\gamma \in (0, \frac{1}{2})$ and $\varepsilon > 2\gamma$. Using Nelson's estimate (B.3) for every $p \geq 2$, the usual Kolmogorov criterion and the embedding $\mathcal{B}_{p,p}^{-\alpha+\frac{2}{p}} \hookrightarrow \mathcal{C}^{-\alpha}$ we finally obtain that

$$\lim_{m\to\infty} \lim_{m'\to\infty} \mathbb{E} \sup_{t\leq T} \| h_m(t) \circ \big(\mathbf{V}_{-\infty}(t) - (\mathbf{I}_{-\infty}(t) \mathbf{I}^m_{-\infty}(t) - \Re_m) \big) \|^p_{\mathcal{C}^{-\alpha}} = 0.$$

Convergence of $h_m \left(\mathbf{V}_{-\infty} - (\mathbf{I}_{-\infty} \mathbf{I}^m_{-\infty} - \Re_m) \right)$ to 0 then follows by Bony estimates (see Proposition A.6). □

For $x \in \mathcal{C}^{-\alpha_0}$, $f \in L^2(\mathbb{R} \times \mathbb{T}^2)$ and $\Re \geq 0$, let $\mathscr{T}(x; f; \Re)$ be the solution map of the equation

$$\begin{aligned} &\big(\partial_t - (\Delta - 1)\big) X = -\sum_{k=0}^{3} a_k \mathcal{H}_k(X, \Re) + f \\ &X|_{t=0} = x \end{aligned} . \tag{4.28}$$

The following proposition is a consequence of Theorem 4.16 and the fact that the solution X to (3.1) is a continuous function of the stochastic objects $\overset{k}{\mathbf{V}}$ (see Definition 3.3) which in turn are continuous functions of the stationary stochastic objects $\overset{k}{\mathbf{V}}_{-\infty}$ (see (2.7)).

Proposition 4.17. *Let $X(\cdot; x)$ be the solution of* (3.1) *for $n = 3$ and $x \in \mathcal{C}^{-\alpha_0}$ and denote by $\mathbb{P}_{X(\cdot;x)}$ its law in $C([0,T]; \mathcal{C}^{-\alpha_0})$. Then*

$$\operatorname{supp} \mathbb{P}_{X(\cdot;x)} = \overline{\{\mathscr{T}(x; f; \Re) : f \in L^2(\mathbb{R} \times \mathbb{T}^2),\ \Re \geq 0\}}^{C([0,T];\mathcal{C}^{-\alpha_0})}.$$

Proof. See the proof of [CF16, Theorem 1.1]. □

We then have the following corollary.

Corollary 4.18. *Let $X(\cdot; x)$ be the solution of* (3.1) *for $n = 3$ and $x \in \mathcal{C}^{-\alpha_0}$. For every $T, \varepsilon > 0$ and $y \in \mathcal{C}^{-\alpha_0}$*

$$\mathbb{P}(X(T; x) \in B(y; \varepsilon)) > 0, \tag{4.29}$$

where $B(y; \varepsilon)$ denotes the open ball of radius ε centred at y in $\mathcal{C}^{-\alpha_0}$.

Proof. It suffices to prove that for every $y \in \mathcal{C}^{\infty}$ there exist $f \in L^2(\mathbb{R} \times \mathbb{T}^2)$ and $\Re \geq 0$ such that $\mathscr{T}(x; f; \Re)(T) = y$. But if we set

$$X(t) = S_1(t)x + \frac{t}{T}\left(y - S_1(T)x\right),$$

for any choice of $\Re \geq 0$ and

$$f(t) = \sum_{k=0}^{3} a_k \mathcal{H}_k(X(t), \Re) + \frac{1}{T}(y - S_1(T)x) - \frac{t}{T}(\Delta - 1)(y - S_1(T)),$$

we have that $X = \mathscr{T}(x; f; \Re)$. Then the result follows by Proposition 4.17 and the fact that $\mathcal{C}^\infty$ is dense in $\mathcal{C}^{-\alpha_0}$. $\square$

4.5 Exponential Mixing

In this section we combine Theorem 4.1, Theorem 4.13 and Corollary 4.18 to prove exponential mixing of the law of the solutions to (3.1) for $n = 3$ with respect to the total variation distance. The restriction $n = 3$ is due to the lack of a general support theorem (see Section 4.4 for details). However, the results of this section can be also used to prove exponential mixing in the case of (3.1) for any $n \geq 3$ odd, given that Corollary 4.18 holds for any $n \geq 3$ odd.

We recall that for any coupling M of probability measures μ_1, μ_2 and F, G measurable functions with respect to the corresponding σ-algebras we have the identity

$$\int (F(x) - G(y))\, M(\mathrm{d}x, \mathrm{d}y) = \int\int (F(x) - G(y))\, \mu_1(\mathrm{d}x)\mu_2(\mathrm{d}y). \tag{4.30}$$

We finally combine the results of the previous sections to prove the following theorem.

Theorem 4.19. *Let $\{P_t : t \geq 0\}$ be the Markov semigroup* (3.28) *of the solution to* (3.1) *for $n = 3$. Then there exists $\lambda \in (0, 1)$ such that*

$$\|P_t^*\delta_x - P_t^*\delta_y\|_{\mathrm{TV}} \leq 1 - \lambda, \tag{4.31}$$

for every $x, y \in \mathcal{C}^{-\alpha_0}$ and $t \geq 3$.

Proof. Let $0 < \alpha < \alpha_0$ and for $R > 0$ consider the subset of $\mathcal{C}^{-\alpha_0}$

$$A_R := \{x \in \mathcal{C}^{-\alpha_0} : \|x\|_{\mathcal{C}^{-\alpha}} \leq R\}$$

which is compact since the embedding $\mathcal{C}^{-\alpha} \hookrightarrow \mathcal{C}^{-\alpha_0}$ is compact (see Proposition A.4). By Theorem 4.13 for every $a \in (0, 1)$ there exists $r \equiv r(a) > 0$ such that for every $x, y \in \bar{B}(0; r)$ and $t \geq 1$

$$\|P_t^*\delta_x - P_t^*\delta_y\|_{\mathrm{TV}} \leq 1 - a.$$

By (4.29) for every $x \in A_R$

$$P_1(x; \bar{B}(0;r)) > 0,$$

which combined with the strong Feller property (which implies the continuity of $P_1(x; A)$ as a function of x for fixed measurable set A) and the fact that A_R is compact implies that there exists $b \equiv b(R) > 0$ such that

$$\inf_{x \in A_R} P_1(x; \bar{B}(0;r)) \geq b.$$

For $t \geq 0$ and $x, y \in A_R \setminus \bar{B}(0;r)$, let $\mathbb{P}_t^{x,y} \in \mathcal{M}_1(\mathcal{C}^{-\alpha_0} \times \mathcal{C}^{-\alpha_0})$ be the product coupling of $P_t(x)$ and $P_t(y)$ given by

$$\mathbb{P}_t^{x,y}(A \times B) = P_t(x; A) P_t(y; B),$$

for every measurable sets $A, B \subset \mathcal{C}^{-\alpha_0}$. Then, for $x, y \in A_R$, $t \geq 2$ and $\Phi \in C_b(\mathcal{C}^{-\alpha_0})$,

$$\begin{aligned} |P_t\Phi(x) - P_t\Phi(y)| &= |\mathbb{E}\left[P_{t-1}\Phi(X(1;x)) - P_{t-1}\Phi(X(1;y))\right]| \\ &= \left| \int \left[P_{t-1}\Phi(\tilde{x}) - P_{t-1}\Phi(\tilde{y})\right] \mathbb{P}_1^{x,y}(\,\mathrm{d}\tilde{x},\,\mathrm{d}\tilde{y}) \right|, \end{aligned}$$

where in the first equality we use the Markov property and in the second (4.30). This implies that

$$\begin{aligned} \|P_t^*\delta_x - P_t^*\delta_y\|_{\mathrm{TV}} &\leq \mathbb{P}_1^{x,y}\left(\left(\bar{B}(0;r) \times \bar{B}(0;r)\right)^c\right) \\ &\quad + (1-a)\mathbb{P}_1^{x,y}\left(\bar{B}(0;r) \times \bar{B}(0;r)\right) \\ &= 1 - a\mathbb{P}_1^{x,y}\left(\bar{B}(0;r) \times \bar{B}(0;r)\right) \\ &\leq 1 - ab^2. \end{aligned}$$

By (4.2) we can choose $R > 0$ sufficiently large such that

$$\inf_{x \in \mathcal{C}^{-\alpha_0}} \inf_{t \geq 1} \mathbb{P}(\|X(t;x)\|_{\mathcal{C}^{-\alpha}} \leq R) > \frac{1}{2}.$$

Then for any $x, y \in \mathcal{C}^{-\alpha_0}$ and $t \geq 3$, using the same coupling argument as above we get

$$\|P_t^*\delta_x - P_t^*\delta_y\|_{\mathrm{TV}} \leq 1 - \frac{ab^2}{4},$$

which completes the proof if we set $\lambda = \frac{ab^2}{4}$. $\square$

The following corollary is the main result of this section and implies exponential mixing.

Corollary 4.20. *There exists a unique invariant measure* $\mu_* \in \mathcal{M}_1(\mathcal{C}^{-\alpha_0})$ *for the semigroup* $\{P_t : t \geq 0\}$ *of the solution to* (3.1) *for* $n = 3$ *such that*

$$\|P_t^* \delta_x - \mu_*\|_{\mathrm{TV}} \leq (1-\lambda)^{\left\lfloor \frac{t}{3} \right\rfloor} \|\delta_x - \mu_*\|_{\mathrm{TV}}, \tag{4.32}$$

for every $x \in \mathcal{C}^{-\alpha_0}$ *and* $t \geq 3$.

Proof. We first notice that for $\mu_1, \mu_2 \in \mathcal{M}_1(\mathcal{C}^{-\alpha_0})$ and every $t \geq 0$ by (4.30) we have that

$$\|P_t^* \mu_1 - P_t^* \mu_2\|_{\mathrm{TV}} \leq \frac{1}{2} \sup_{\|\Phi\|_\infty \leq 1} \int \int |P_t \Phi(x) - P_t \Phi(y)| \, M(\,\mathrm{d}x, \,\mathrm{d}y),$$

for any coupling $M \in \mathcal{M}_1(\mathcal{C}^{-\alpha_0} \times \mathcal{C}^{-\alpha_0})$ of μ_1 and μ_2. Thus by (4.31) for $t \geq 3$

$$\|P_t^* \mu_1 - P_t^* \mu_2\|_{\mathrm{TV}} \leq (1-\lambda) \left(1 - M(\{(x,x) : x \in \mathcal{C}^{-\alpha_0}\})\right)$$

and using the characterisation of the total variation distance given by

$$\|\mu_1 - \mu_2\|_{\mathrm{TV}} = \inf \left\{1 - M(\{(x,x) : x \in \mathcal{C}^{-\alpha_0}\}) : M \text{ coupling of } \mu_1 \text{ and } \mu_2\right\}$$

we get that

$$\|P_t^* \mu_1 - P_t^* \mu_2\|_{\mathrm{TV}} \leq (1-\lambda) \, \|\mu_1 - \mu_2\|_{\mathrm{TV}}.$$

This implies that $\{P_t : t \geq 0\}$ has a unique invariant measure $\mu_* \in \mathcal{M}_1(\mathcal{C}^{-\alpha_0})$, since by Proposition [DPZ96, Proposition 3.2.5] any two distinct invariant measures are singular. Finally, for $x \in \mathcal{C}^{-\alpha_0}$ and $t \geq 3$

$$\|P_t^* \delta_x - \mu_*\|_{\mathrm{TV}} \leq (1-\lambda) \|P_{t-3}^* \delta_x - \mu_*\|_{\mathrm{TV}},$$

which implies (4.32). □

Chapter 5

Metastability

5.1 Introduction

In this chapter we study the behaviour of solutions to the 2-dimensional Allen–Cahn equation, perturbed by a small noise term, given by

$$(\partial_t - \Delta)X = -X^3 + X + \sqrt{2\varepsilon}\xi, \tag{5.1}$$

for a small parameter $\varepsilon > 0$. The deterministic equation is given by

$$(\partial_t - \Delta)X = -X^3 + X, \tag{5.2}$$

and it is well-known that (5.2) is a gradient flow with respect to the double-well potential

$$V(X) := \int \left(\frac{1}{2}|\nabla X(z)|^2 - \frac{1}{2}|X(z)|^2 + \frac{1}{4}|X(z)|^4\right) \mathrm{d}z. \tag{5.3}$$

In the case $d = 1$, where the solution X depends on time and a 1-dimensional spatial argument, the behaviour of solutions to (5.1) is well-understood. They exhibit the phenomenon of metastability, that is, they typically spend large stretches of time close to the minimisers of the potential (5.3) with rare and relatively quick noise-induced transitions between them. Early contributions go back to the 80s where Faris and Jona–Lasinio [FJL82] studied the system on the level of large deviations.

We are particularly interested in the "exponential loss of memory property" first observed by Martinelli, Olivieri and Scoppola in [MS88, MOS89]. They studied the flow map induced by (5.1), that is, the random map $x \mapsto X(t;x)$ which associates to any initial condition the corresponding solution at time t, and showed that for large values of t the map essentially becomes constant. They also showed that with overwhelming probability, solutions that start within the basin of attraction of the same minimiser of V contract exponentially fast, with exponential rate given by the smallest eigenvalue of the linearisation of V in this minimiser. This implies for example that the law of such solutions at large times is essentially insensitive to the precise location at which they are started.

It is very natural to study this behaviour in higher dimensions, but as we already discussed in Section 1.1 when $d \geq 2$, equation (5.1) is ill-posed. In particular, solutions have to be interpreted in the sense of Schwartz distributions and one has to work with the renormalised equation formally given by

$$(\partial_t - \Delta)X = -X^3 + (1 + 3\varepsilon\infty)X + \sqrt{2\varepsilon}\xi. \tag{5.4}$$

Note that formally, this renormalisation corresponds to moving the minima of the double-well potential out to $\pm\infty$ and making them infinitely deep at the same time. So at first glance, it seems unclear why these renormalised distribution-valued solutions should exhibit similar behaviour to the 1-dimensional function-valued solutions of (5.1).

In [HW15] the authors studied the small ε asymptotics for (5.4) in $d = 2$ and 3 on the level of Freidlin-Wentzell type large deviations. They obtained a large deviation principle with rate function $\mathcal{I}$ given by

$$\mathcal{I}(X) := \frac{1}{4}\int_0^T \int \Big(\partial_t X(t,z) - \big(\Delta X(t,z) - \big(X(t,z)^3 - X(t,z)\big)\big)\Big)^2 \,\mathrm{d}z\,\mathrm{d}t. \tag{5.5}$$

In fact, a result in a similar spirit had already appeared in the 90s [JLM90]. The striking fact is that this rate function is exactly the 2-dimensional version of the rate function obtained in the 1-dimensional case [FJL82]; the infinite renormalisation constant does not affect the rate functional. This result implies that for small ε, solutions of the renormalised SPDE (5.4) stay close to solutions of the deterministic PDE (5.2) suggesting that (5.4) may indeed be the natural small noise perturbation of (5.2).

Here we consider (5.4) over a 2-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/L\mathbb{Z}^2$ for $L < 2\pi$. It is known that under this assumption on the torus size L, the deterministic equation (5.2) has exactly 3 stationary solutions, namely the constant profiles $-1, 0, 1$ (see [KORVE07, Appendix B.1]). The profiles ± 1 are stable minimisers of V and the profile 0 is unstable. We prove that in the small noise regime, solutions that start close to the same stable minimiser ± 1 contract exponentially fast with overwhelming probability. The exponential contraction rate is arbitrarily close to 2, the second derivative of the double-well $x \mapsto \frac{1}{4}x^4 - \frac{1}{2}x^2$ in ± 1. This is precisely the 2-dimensional version of [MOS89, Corollary 3.1].

On a technical level we work with the Da Prato–Debussche decomposition discussed in Chapter 3. An immediate observation is that differences of any two profiles have much better regularity than the solutions themselves. We split the time axis into random "good" and "bad" intervals depending on whether a reference profile is close to ± 1 or not. The key idea is that on "good" intervals solutions should contract exponentially, while they should not diverge too fast on "bad" intervals. Furthermore, "good" intervals should be much longer than "bad" intervals.

The control on the "good" intervals is relatively straightforward: the exponential contraction follows by linearising the equation and the fact that these intervals are typically long follows from exponential moment bounds of the stochastic objects appearing in the Da Prato–Debussche decomposition (see Proposition 5.8). The control on the "bad" intervals is much more involved; in the 1-dimensional case two profiles cannot diverge too fast, because the second derivative of the double-well potential is bounded from below. But in the 2-dimensional case, where solutions are distribution-valued, there is no obvious counterpart of this property. Instead we use the strong a priori estimate in Proposition 3.10 and the local Lipschitz continuity of the non-linearity. Ultimately, this yields an exponential growth bound where the exponential rate is given by a polynomial in the explicit stochastic objects. We use a large deviation estimate to prove that these intervals cannot be too long (see Proposition 5.10). In the final step we show that the exponential contraction holds for all t if a certain random walk with positive drift stays positive for all times. This random walk is then analysed using techniques developed for the classical Cramér–Lundberg model in risk theory (see Proposition 5.13).

As a corollary of this theorem we prove an Eyring–Kramers law for the transition times of X. In [BDGW17] the authors studied spectral Galerkin approximations X^N of (5.4) and obtained explicit estimates on the expected first transition times from a neighbourhood of -1 to a neighbourhood of 1. These estimates give the precise asymptotics as $\varepsilon \to 0$ and hold uniformly in the discretisation parameter N. Their method was based on the potential theoretic approach developed in the finite-dimensional context by Bovier et al. in [BEGK04]. This approach relies heavily on the reversibility of the dynamics and provides explicit formulas for the expected transition times in terms of certain integrals of the reversible measure. The key observation in [BDGW17] was that in the context of (5.1) these integrals can be analysed uniformly in the parameter N using the classical Nelson's estimate [Nel73] from constructive Quantum Field Theory. However, the result in [BDGW17] was not optimal for the following two reasons: First, it does not allow to pass to the limit as $N \to \infty$ to retrieve the estimate for the transition times of X. Second, and more important, the bounds could only be obtained for a certain N-dependent choice of initial distribution on the neighbourhood of -1. This problem is inherent to the potential theoretic approach, which only yields an exact formula for the diffusion started in this so-called *normalised equilibrium measure*. In fact, a large part of the original work [BEGK04] was dedicated to removing this problem using regularity theory for the finite-dimensional transition probabilities.

Here, we overcome these two barriers. We first justify the passage to the limit $N \to \infty$ based on results discussed in Section 1.3.1: we use the strong dissipative bound (Theorem 1.4) on the level of the approximation X^N (see Proposition 5.33) and the support Theorem 1.6 to prove uniform integrability of the transition times of X^N. The only difficulty here comes from the action of the Galerkin projection on the non-linearity which does not allow to test the equation with powers greater than 1. To remove the unnatural assumption on the initial distribution we make use of the exponential contraction estimate, Theorem 1.9. This estimate allows us to couple the solution started with an arbitrary but fixed initial condition with the solution started in the normalised equilibrium measure.

5.1.1 Outline

In Section 5.2 we prove the "exponential loss of memory property" for the solution of (5.4) (as defined by the equivalent of Definition 3.3 when ξ is replaced by

$\sqrt{\varepsilon}\xi$). In Section 5.3 we prove an Eyring–Kramers law for the transition times between the minimisers of the potential (5.3).

5.1.2 Notation

Following the equivalent of Definition 3.3 in the case of (5.4), we write $X(\cdot;x) = v(\cdot;x) + \varepsilon^{\frac{1}{2}} \uparrow(\cdot)$, where $\uparrow$ is defined in (2.3) for $s = 0$ and the remainder term $v(\cdot;x)$ solves

$$\begin{aligned} &(\partial_t - \Delta)\, v = -v^3 + v - \left(3v^2 \varepsilon^{\frac{1}{2}} \uparrow + 3v\varepsilon \curlyvee + \varepsilon^{\frac{3}{2}} \Psi - 2\varepsilon^{\frac{1}{2}} \uparrow\right), \\ &v\big|_{t=0} = x \end{aligned} \tag{5.6}$$

where $\curlyvee$, Ψ are the 2-nd and 3-rd Wick powers of $\uparrow$ defined in (2.7) for $s = 0$. We furthermore impose that $\mathbb{T}^2 = \mathbb{R}^2 / L\mathbb{Z}^2$, for $L < 2\pi$.

Remark 5.1. To ease the notation we hide the dependence of X and v on the parameter ε. However in this chapter both objects should be thought of as being indexed by ε.

By Theorem 2.1 and Proposition 2.2 the stochastic objects $\uparrow$, $\curlyvee$ and Ψ can be realised as continuous processes taking values in $\mathcal{C}^{-\alpha}$ for $\alpha > 0$, such that $\mathbb{P}$-almost surely for every $T > 0$, and $\alpha' > 0$

$$\max \left\{ \sup_{t \le T} \|\uparrow(t)\|_{\mathcal{C}^{-\alpha}},\ \sup_{t \le T} (t \wedge 1)^{\alpha'} \|\curlyvee(t)\|_{\mathcal{C}^{-\alpha}},\ \sup_{t \le T} (t \wedge 1)^{2\alpha'} \|\Psi(t)\|_{\mathcal{C}^{-\alpha}} \right\} < \infty. \tag{5.7}$$

Throughout this section we use $\overset{n}{\nabla}$ to refer to all the stochastic objects $\uparrow$, $\curlyvee$ and Ψ simultaneously. In this notation (5.7) turns into

$$\sup_{t \le T} (t \wedge 1)^{(n-1)\alpha'} \|\overset{n}{\nabla}(t)\|_{\mathcal{C}^{-\alpha}} < \infty.$$

As in Chapter 3 (see (3.5)), we fix $\alpha_0 \in (0, \frac{1}{3})$ (to measure the regularity of the initial condition x in $\mathcal{C}^{-\alpha_0}$), $\beta > 0$ (to measure the regularity of v in $\mathcal{C}^{\beta}$) and $\gamma > 0$ (to measure the rate of blow-up of $\|v(t;x)\|_{\mathcal{C}^{\beta}}$ for t close to 0) such that

$$\gamma < \frac{1}{3}, \quad \frac{\alpha_0 + \beta}{2} < \gamma. \tag{5.8}$$

We also assume that $\alpha' > 0$ and $\alpha > 0$ in (5.7) satisfy

$$\alpha' < \gamma, \quad \alpha < \alpha_0, \quad \frac{\alpha + \beta}{2} + 2\gamma < 1. \tag{5.9}$$

By Theorems 3.6 and 3.12 for every $x \in \mathcal{C}^{-\alpha_0}$ there exist a unique solution $v \in C\left((0,\infty);\mathcal{C}^{\beta}\right)$ of (5.6) such that for every $T > 0$

$$\sup_{t \le T} (t \wedge 1)^{\gamma} \|v(t;x)\|_{\mathcal{C}^{\beta}} < \infty.$$

Remark 5.2. In Condition (5.8) β has to be strictly less than $\frac{2}{3}$. This is necessary if one wants to treat all of the terms arising in a fixed point problem for (5.6) with the same norm for v. A simple post-processing of Theorems 3.6 and 3.12 shows that in fact v is continuous in time taking values in $\mathcal{C}^{2-\lambda}$ for any $\lambda > \alpha$.

Equations (2.2), (5.6) suggest that indeed X can be seen as a perturbation of the Allen-Cahn equation (5.2), because the terms $\mathbf{I}$, $\mathbf{V}$ and $\mathbf{\Psi}$ in (5.6) all appear with a positive power of ε. It is important to note that v is much more regular than X. The irregular part of $X(\cdot;x)$ is $\varepsilon^{\frac{1}{2}}\mathbf{I}$. Therefore differences of solutions are much more regular than solutions themselves.

We repeatedly work with the restarted stochastic terms $\mathbf{I}_s$, $\mathbf{V}_s$ and $\mathbf{\Psi}_s$ define in (2.7). By Proposition 2.3 for every $s > 0$, $\mathbf{V}^{(n)}_s(s+\cdot)$ is independent of $\mathcal{F}_s$ and equal in law to $\mathbf{V}^{(n)}(\cdot)$. For $t \ge s$ we can define a restarted remainder $v_s(t;X(s;x))$ through the identity $X(t;x) = v_s(t;X(s;x)) + \varepsilon^{\frac{1}{2}}\mathbf{I}_s(t)$. Rearranging (5.6) and using the pathwise identities in Corollary 2.4 one can see that v_s solves

$$\begin{aligned} (\partial_t - \Delta)\, v_s &= -v_s^3 + v_s - \left(3v_s^2\varepsilon^{\frac{1}{2}}\mathbf{I}_s + 3v_s\varepsilon\mathbf{V}_s + \varepsilon^{\frac{3}{2}}\mathbf{\Psi}_s - 2\varepsilon^{\frac{1}{2}}\mathbf{I}_s\right) \\ v_s\big|_{t=s} &= X(s;x) \end{aligned}. \tag{5.10}$$

Finally, for any Banach space $(V, \|\cdot\|_V)$ we denote by $B_V(x_0;\delta)$ the open ball $\{x \in V : \|x - x_0\|_V < \delta\}$ and by $\bar{B}_V(x_0;\delta)$ its closure.

5.2 Exponential Loss of Memory

In this section we prove the following theorem. From now on whenever we write ± 1 we simply mean that the statement holds for -1 and 1 separately.

Theorem 5.3. *For every $\kappa > 0$ there exist $\delta_0, a_0, C > 0$ and $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \le \varepsilon_0$*

$$\inf_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0} \mathbb{P}\left(\sup_{\|y-x\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0} \frac{\|X(t;y) - X(t;x)\|_{\mathcal{C}^{\beta}}}{\|y - x\|_{\mathcal{C}^{-\alpha_0}}} \le C\mathrm{e}^{-(2-\kappa)t},\ \forall t \ge 1 \right)$$
$$\ge 1 - \mathrm{e}^{-a_0/\varepsilon}.$$

Proof. See Section 5.2.1. □

This theorem is a variant of [MOS89, Corollary 3.1] in space dimension $d = 2$. There the supremum is taken over both x and y inside the probability measure. We also obtain this version of the theorem as a corollary.

Corollary 5.4. *For every $\kappa > 0$ there exist $\delta_0, a_0, C > 0$ and $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \le \varepsilon_0$*

$$\mathbb{P}\left(\sup_{x,y \in \bar{B}_{\mathcal{C}^{-\alpha_0}}(\pm 1;\delta_0)} \frac{\|X(t;y) - X(t;x)\|_{\mathcal{C}^\beta}}{\|y-x\|_{\mathcal{C}^{-\alpha_0}}} \le C\mathrm{e}^{-(2-\kappa)t},\ \forall t \ge 1 \right) \ge 1 - \mathrm{e}^{-a_0/\varepsilon}.$$

Proof. See Section 5.2.1. □

Remark 5.5. The restriction $t \ge 1$ in Theorem 5.3 appears only because we measure $y - x$ in a lower regularity norm than $X(t;y) - X(t;x)$. To prove the theorem we first prove Theorem 5.15 were we assume that $y - x \in \mathcal{C}^\beta$ and in this case we prove a bound which holds for every $t > 0$.

Remark 5.6. Theorem 5.3 is an asymptotic coupling of solutions that start close to the same minimiser. In [MOS89, Proposition 3.4] it was shown that in the 1-dimensional case, solutions which start with initial conditions x and y close to different minimisers also contract exponentially fast, but only after time $T_\varepsilon \propto \mathrm{e}^{[(V(0)-V(\pm 1))+\eta]/\varepsilon}$ for any $\eta > 0$. This is the "typical" time needed for one of the two profiles to jump close to the other minimiser. We expect that Theorem 5.3 and the large deviation theory developed in [HW15] could be combined to prove a similar result in the case $d = 2$.

We now define two sequences $\{\nu_i(x)\}_{i\ge 1}$ and $\{\rho_i(x)\}_{i \ge 1}$ of stopping times which partition our time axis and allow us to keep track of the time spent close to and away from the minimisers ± 1 (see Figure 5.1 for a sketch). On the "good" intervals $[\rho_{i-1}(x), \nu_i(x)]$ we require both the restarted diagrams $\bigvee^{n}_{\rho_{i-1}(x)}$ to be small and the restarted remainder $v_{\rho_{i-1}(x)}$ to be close to ± 1. The "bad" intervals $[\nu_i(x), \rho_i(x)]$ end when $X(\cdot;x)$ re-enters a small neighbourhood of the minimisers. The stopping times $\rho_i(x)$ are defined in terms of the $\mathcal{C}^{-\alpha_0}$ norm for $X(\cdot;x)$, while we define good intervals in terms of the stronger $\mathcal{C}^\beta$ topology for $v_{\rho_{i-1}(x)}$. To connect the two, we need to allow for a blow-up close to the starting point of the "good" intervals.

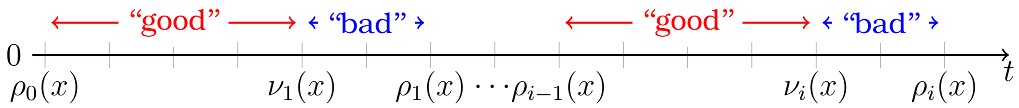

Figure 5.1: A partition of the time axis with respect to the times $\nu_i(x)$ and $\rho_i(x)$. The "good" intervals are "typically" much larger than the "bad" intervals.

Definition 5.7. For $x \in \mathcal{C}^{-\alpha_0}$ we define the stopping times $\{\rho_i(x)\}_{i\geq 0}$, $\{\nu_i(x)\}_{i\geq 1}$ recursively by $\rho_0(x) = 0$ and

$$\begin{aligned}\nu_i(x) := \inf \Big\{ t > \rho_{i-1}(x) : ((t-\rho_{i-1}(x)) \wedge 1)^{(n-1)\alpha'} \|\varepsilon^{\frac{n}{2}} \overset{n}{\vee}_{\rho_{i-1}(x)}(t)\|_{\mathcal{C}^{-\alpha}} \geq \delta_2^n \\ \text{or} \min_{x_* \in \{-1,1\}} ((t-\rho_{i-1}(x)) \wedge 1)^{\gamma} \|v_{\rho_{i-1}(x)}(t; X(\rho_i(x);x)) - x_*\|_{\mathcal{C}^{\beta}} \geq \delta_1 \Big\} \\ \rho_i(x) := \inf\{t > \nu_i(x) : \min_{x_* \in \{-1,1\}} \|X(t;x) - x_*\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0\}.\end{aligned}$$

We now define the time increments

$$\begin{aligned}\tau_i(x) &= \nu_i(x) - \rho_{i-1}(x). \\ \sigma_i(x) &= \rho_i(x) - \nu_i(x).\end{aligned} \tag{5.11}$$

The process $X(\cdot;x)$ is expected to spend long time intervals close to the minimisers ± 1, which corresponds to large values of $\tau_i(x)$. Large values of $\sigma_i(x)$ are "atypical". This behaviour is established Propositions 5.19 and 5.22.

The following proposition shows contraction on the "good" intervals. We distinguish between the cases (5.12) and (5.13) for $y - x$ that lie in $\mathcal{C}^{\beta}$ and $\mathcal{C}^{-\alpha_0}$ respectively. The Da Prato–Debussche decomposition shows that differences of any two profiles lie in $\mathcal{C}^{\beta}$ for any $t > 0$ but at $t = 0$ they maintain the irregularity of the initial conditions. Hence we only use (5.13) on the first "good" interval.

Proposition 5.8. *For every $\kappa > 0$ there exist $\delta_0, \delta_1, \delta_2 > 0$ and $C > 0$ such that if $\|x - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0$ and $y - x \in \mathcal{C}^{\beta}$, $\|y - x\|_{\mathcal{C}^{\beta}} \leq \delta_0$ then*

$$\|X(t;y) - X(t;x)\|_{\mathcal{C}^{\beta}} \leq C \exp\left\{-\left(2 - \frac{\kappa}{2}\right) t\right\} \|y - x\|_{\mathcal{C}^{\beta}} \tag{5.12}$$

for every $t \leq \tau_1(x)$ defined with respect to δ_1 and δ_2. If we only assume that $\|y - x\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0$ then

$$(t \wedge 1)^{\gamma} \|X(t;y) - X(t;x)\|_{\mathcal{C}^{\beta}} \leq C \exp\left\{-\left(2 - \frac{\kappa}{2}\right) t\right\} \|y - x\|_{\mathcal{C}^{-\alpha_0}} \tag{5.13}$$

for every $t \leq \tau_1(x)$.

Proof. See Section 5.2.2.1. □

Our next aim is to control the growth of the differences on the "bad" intervals in terms of the stochastic objects $\triangledown^{n}$. This is done by partitioning the intervals $[\nu_i(x), \rho_i(x)]$ into tiles of length one. To achieve independence we restart the stochastic objects at the starting point of each tile.

Definition 5.9. For $k \geq 0$ and $\rho \geq \nu \geq 0$ let $t_k = \nu + k$. For $k \geq 1$ we define a random variable $L_k(\nu, \rho)$ by

$$L_k(\nu, \rho) := \left(\sup_{t \in [t_{k-1}, t_k \wedge \rho]} (t - t_{k-1})^{(n-1)\alpha'} \| \varepsilon^{\frac{n}{2}} \triangledown^{n}_{t_{k-1}}(t) \|_{\mathcal{C}^{-\alpha}} \right)^{\frac{2}{n}}. \tag{5.14}$$

In our analysis we use a second tiling defined by setting $s_k = t_k + \frac{1}{2}$, i.e. the tiles $[t_k, t_{k+1}]$ and $[s_k, s_{k+1}]$ overlap. In order to bound $X(t; y) - X(t; x)$ on a time interval $[t_k, s_k]$ we restart the stochastic objects at s_{k-1} and write $X(t; y) - X(t; x) = v_{s_{k-1}}(t; X(s_{k-1}; y)) - v_{s_{k-1}}(t; X(s_{k-1}; x))$. In Lemma 5.16 we upgrade the a priori estimates obtained in Proposition 3.10 to get a control on the $\mathcal{C}^\beta$ norm of both remainders. This bound holds uniformly over all possible values of $X(s_{k-1}; y)$ and $X(s_{k-1}; x)$ and while the bound allows for a blow-up for times t close to s_{k-1} it holds uniformly over all times in $[t_k, s_k]$. Ultimately, the bound only depends on $L_k(\nu + \frac{1}{2}, \rho)$ in a polynomial way as shown in Figure 5.2. Then we can use the local Lipschitz property of the non-linearity in (5.6) to bound the exponential growth rate of $X(t; y) - X(t; x)$. For the first interval $[t_0, t_1]$ we do not use this trick, because we want to avoid bounds that depend on the realisation of the white noise outside of $[\nu, \rho]$. On this interval, we make use of an a priori assumption that we have some control on $\|X(\nu; y)\|_{\mathcal{C}^{-\alpha_0}}$ and $\|X(\nu; x)\|_{\mathcal{C}^{-\alpha_0}}$.

Proposition 5.10. *Let $R > 0$. Then there exists a constant $C \equiv C(R) > 0$ such that for every $\|X(\nu; x)\|_{\mathcal{C}^{-\alpha_0}}, \|X(\nu; y)\|_{\mathcal{C}^{-\alpha_0}} \leq R$, $\rho > \nu \geq 0$ and $t \in [\nu, \rho]$*

$$\|X(t; y) - X(t; x)\|_{\mathcal{C}^\beta} \leq C \exp\{L(\nu, \rho; t - \nu)\} \|X(\nu; y) - X(\nu; x)\|_{\mathcal{C}^\beta}, \tag{5.15}$$

where

$$L(\nu, \rho; t - \nu) = \frac{c_0}{2} \sum_{k=1}^{\lfloor t-\nu \rfloor} \sum_{l=0,\frac{1}{2}} (1 \vee L_k(\nu + l, \rho))^{p_0} + L_0(t - \nu) \tag{5.16}$$

for L_k as in (5.14), *and for some constants $p_0 \geq 1$ and $c_0 \equiv c_0(R), L_0 \equiv L_0(R) \geq 0$.*

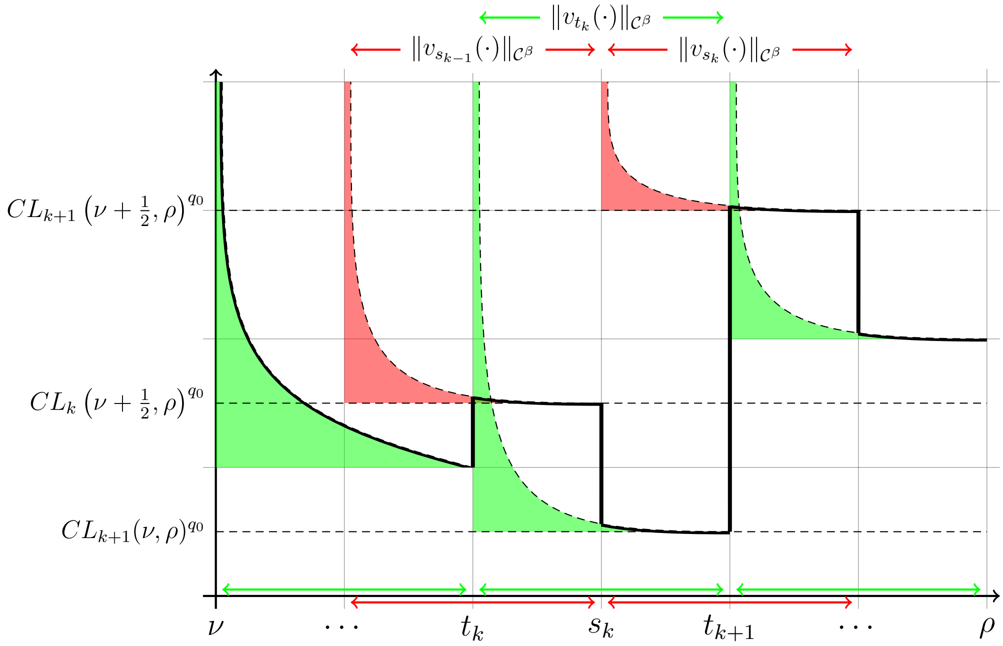

Figure 5.2: Bounds on the $\mathcal{C}^\beta$ norm of the restarted remainder v on the overlapping tiles of the partition of $[\nu,\rho]$. On a time interval $[t_k, s_k]$ we restart the stochastic objects at time s_{k-1} and bound $v_{s_{k-1}}$ by a polynomial function of $L_k\left(\nu+\frac{1}{2},\rho\right)$. On a time interval $[s_k, t_{k+1}]$ we restart the stochastic objects at time t_k and bound v_{t_k} by a polynomial function of $L_k(\nu,\rho)$.

Proof. See Section 5.2.2.2. □

If we assume that $y-x\in\mathcal{C}^\beta$, combining the estimates in Propositions 5.8 and 5.10 suggest the bound

$$\begin{aligned}&\|X(\rho_N(x);y)-X(\rho_N(x);x)\|_{\mathcal{C}^\beta}\\ &\quad\le\exp\left\{\sum_{i\le N}\left[-\left(2-\frac{\kappa}{2}\right)\tau_i(x)+L(\nu_i(x),\rho_i(x);\sigma_i(x))+2\log C\right]\right\}\\ &\qquad\times\|y-x\|_{\mathcal{C}^\beta},\end{aligned}\tag{5.17}$$

for any $N\ge1$. If we can show that the exponents satisfy

$$\sum_{i\le N}\left[-\left(2-\frac{\kappa}{2}\right)\tau_i(x)+L(\nu_i(x),\rho_i(x);\sigma_i(x))+2\log C\right]\le-(2-\kappa)\rho_N(x),$$

then (5.17) yields exponential contraction at time $\rho_N(x)$ with rate $2-\kappa$. The difference of the right hand side and the left hand side of the last inequality is given by the random walk $S_N(x)$ in the next definition.

Definition 5.11. For $\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0$ we define the random walk $(S_N(x))_{N\geq 1}$ by

$$S_N(x) := \sum_{i\leq N} \Big[\frac{\kappa}{2}\tau_i(x) - \big(L(\nu_i(x), \rho_i(x); \sigma_i(x)) + (2-\kappa)\sigma_i(x) + M_0\big)\Big]$$

where $M_0 = 2\log C$ for $C > 0$ as in Propositions 5.8 and 5.10.

The next proposition shows that the random walk $S_N(x)$ stays positive for every $N \geq 1$ with overwhelming probability (see Figure 5.3 for an illustration). The proof is based on a variant of the classical Cramér–Lundberg model in risk theory (see [EKM97, Chapter 1.2]). In this classical model a random walk $S_N = \sum_{i\leq N}(f_i - g_i)$ with i.i.d. exponential random variables f_i and i.i.d. non-negative random variables g_i is considered. The probability for S_N to stay positive for every $N \geq 1$ can be calculated explicitly in terms of the expectations of f_i and g_i using a renewal equation. In our case we use the Markov property and Propositions 5.19 and Proposition 5.23 to compare the random walk $S_N(x)$ in Definition 5.11 to this classical case.

Remark 5.12. If the family $\{L(\nu_i(x), \rho_i(x); \sigma_i(x)) + (2-\kappa)\sigma_i(x) + M_0\}_{i\geq 1}$ had exponential moments, a simple exponential Chebyshev argument would imply the following proposition without any reference to the Cramér–Lundberg model. However, by (5.14) and (5.16) one sees that $L(\nu_i(x), \rho_i(x); \sigma_i(x))$ is a polynomial of potentially high degree in the explicit stochastic objects (which are themselves polynomials of the Gaussian noise ξ). Hence, we cannot expect more than stretched exponential moments, and indeed, such bounds are established in Proposition 5.23. In the proof of the next proposition we also use an exponential Chebyshev argument, but only to compare $\frac{\kappa}{2}\tau_i(x)$ with a suitable exponential random variable which does not depend on x.

Proposition 5.13. *For every $\kappa > 0$ there exist $a_0 > 0$ and $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \leq \varepsilon_0$*

$$\inf_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}}\leq\delta_0} \mathbb{P}(S_N(x) \geq 0 \text{ for every } N \geq 1) \geq 1 - \mathrm{e}^{-a_0/\varepsilon}. \tag{5.18}$$

Proof. See Section 5.2.3.3. ☐

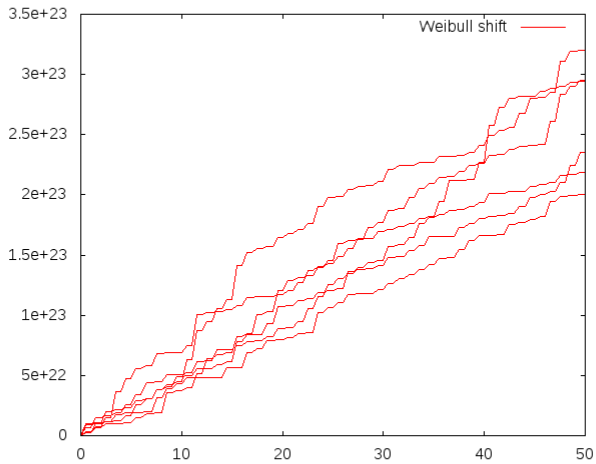

Figure 5.3: "Typical" realisations of a random walk $S_N = \sum_{i \le N}(f_i - g_i)$ for $f_i \sim \mathrm{e}^{0.5/\varepsilon}\exp(1)$, $g_i \sim \mathrm{e}^{0.1/\varepsilon}\mathrm{Weibull}(0.5, 1)$, $N = 50$ and $\varepsilon = 0.01$. The choice of a Weibull distribution here captures the fact that the random variables $L(\nu_i(x), \rho_i(x); \sigma_i(x)) + (2-\kappa)\sigma_i(x) + M_0$ in Definition 5.11 have stretched exponential tails as shown in Proposition 5.23.

5.2.1 Proof of Theorem 5.3

We first treat the case where $y - x \in \mathcal{C}^\beta$. Let $x \in \mathcal{C}^{-\alpha_0}$ such that $\|x - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0$ and let y be such that $y - x \in \mathcal{C}^\beta$ and $\|y - x\|_{\mathcal{C}^\beta} \le \delta_0$. We also write $Y(t) = X(t; y) - X(t; x)$. We consider the event

$$\mathcal{S}(x) = \{S_N(x) \ge 0 \text{ for every } N \ge 1\} \tag{5.19}$$

for $S_N(x)$ as in Definition 5.11.

We first prove the following proposition which provides explicit estimates on the differences at the stopping times $\nu_N(x)$ and $\rho_N(x)$ for every $N \ge 1$ and $\omega \in \mathcal{S}(x)$ by iterating Propositions 5.8 and 5.10. To shorten the notation we drop the explicit dependence on the starting point x in the stopping times ν_N and ρ_N and the random walk S_N. We also drop the dependence on the realisation ω but we assume throughout that $\omega \in \mathcal{S}(x)$.

Proposition 5.14. *For any $\kappa > 0$ let $C > 0$ be as in Proposition 5.8. Then for every $\omega \in \mathcal{S}(x)$ and $N \ge 1$*

$$\|Y(\nu_N)\|_{\mathcal{C}^\beta} \le C \exp\left\{-S_{N-1} - \frac{\kappa}{2}\tau_N\right\} \exp\{-(2-\kappa)\nu_N\} \|Y(0)\|_{\mathcal{C}^\beta} \tag{5.20}$$

$$\|Y(\rho_N)\|_{\mathcal{C}^\beta} \le \exp\{-S_N\} \exp\{-(2-\kappa)\rho_N\} \|Y(0)\|_{\mathcal{C}^\beta}. \tag{5.21}$$

Proof. We prove our claim by induction on $N \geq 1$, observing that it is obvious for $N = 0$.

To prove (5.20) for $N+1$ we first notice that by the definition of ρ_N we have that $\|X^\varepsilon(\rho_N; x) - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0$ and since $\omega \in \mathcal{S}(x)$ (5.21) implies that $\|Y(\rho_N)\|_{\mathcal{C}^\beta} \leq \delta_0$. Hence we can use (5.12) to get

$$\|Y(\nu_{N+1})\|_{\mathcal{C}^\beta} \lesssim \exp\left\{-\frac{\kappa}{2}\tau_{N+1}\right\} \exp\left\{-(2-\kappa)\tau_{N+1}\right\} \|Y(\rho_N)\|_{\mathcal{C}^\beta}.$$

Combining with the estimate on $\|Y(\rho_N)\|_{\mathcal{C}^\beta}$ the above implies (5.20) for $N+1$.

To prove (5.21) for $N+1$ we first notice that by Proposition 5.18

$$\|X(\nu_{N+1}; x)\|_{\mathcal{C}^{-\alpha_0}} \leq 2\delta_1 + 1.$$

This bound, (5.20) for $N+1$ and the triangle inequality imply that

$$\|X(\nu_{N+1}; y)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0 + 2\delta_1 + 1.$$

Hence, we can use Proposition 5.10 for $\nu = \nu_{N+1}$, $\rho = \rho_{N+1}$ and $R = \delta_0 + 2\delta_1 + 1$ to obtain

$$\|Y(\rho_{N+1})\|_{\mathcal{C}^\beta} \lesssim \exp\left\{L(\nu_{N+1}, \rho_{N+1}; \sigma_{N+1})\right\} \|Y(\nu_{N+1})\|_{\mathcal{C}^\beta}.$$

If we combine with (5.20) for $N+1$ we have that

$$\begin{aligned}\|Y(\rho_{N+1})\|_{\mathcal{C}^\beta} \leq \exp&\left\{L(\nu_{N+1}, \rho_{N+1}; \sigma_{N+1}) + M_0\right\} \exp\left\{-S_N\right\} \\ &\times \exp\left\{-\frac{\kappa}{2}\tau_{N+1}\right\} \exp\left\{-(2-\kappa)\nu_{N+1}\right\} \|Y(0)\|_{\mathcal{C}^\beta}.\end{aligned}$$

We then rearrange the terms to obtain (5.21), which completes the proof. $\square$

We are ready to prove the following version of Theorem 5.3 for sufficiently smooth initial conditions.

Theorem 5.15. *For every $\kappa > 0$ there exist $\delta_0, a_0, C > 0$ and $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \leq \varepsilon_0$*

$$\begin{aligned}\inf_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0} \mathbb{P}&\left(\sup_{\substack{y-x\in\mathcal{C}^\beta \\ \|y-x\|_{\mathcal{C}^\beta}\leq\delta_0}} \frac{\|X(t;y) - X(t;x)\|_{\mathcal{C}^\beta}}{\|y-x\|_{\mathcal{C}^\beta}} \leq C\mathrm{e}^{-(2-\kappa)t}, \ \forall t \geq 0\right) \\ &\geq 1 - \mathrm{e}^{-a_0/\varepsilon}.\end{aligned}$$

Proof. Let $\omega \in \mathcal{S}(x)$ as in (5.19). For any $t > 0$ there exists $N \equiv N(\omega) \geq 0$ such that $t \in [\rho_N, \nu_{N+1})$ or $t \in [\nu_{N+1}, \rho_{N+1})$.

If $t \in [\rho_N, \nu_{N+1})$ then

$$\begin{aligned}
&\|X(t;y) - X(t;x)\|_{\mathcal{C}^\beta} \\
&\overset{(5.12),(5.21)}{\lesssim} \exp\left\{-\left(2-\frac{\kappa}{2}\right)(t-\rho_N)\right\} \|X(\rho_N;y)) - X(\rho_N;x)\|_{\mathcal{C}^\beta} \\
&= \exp\left\{-\frac{\kappa}{2}(t-\rho_N)\right\} \exp\left\{-(2-\kappa)(t-\rho_N)\right\} \|X(\rho_N;y)) - X(\rho_N;y)\|_{\mathcal{C}^\beta} \\
&\overset{(5.21)}{\lesssim} \exp\left\{-(2-\kappa)t\right\} \|y-x\|_{\mathcal{C}^\beta}.
\end{aligned}$$

If $t \in [\nu_{N+1}, \rho_{N+1})$ then

$$\begin{aligned}
\|X(t;y) - X(t;x)\|_{\mathcal{C}^\beta} &\overset{(5.15)}{\lesssim} \exp\{L(\nu_{N+1}, \rho_{N+1}; t-\nu_{N+1})\} \\
&\quad \times \|X(\nu_{N+1};y) - X(\nu_{N+1};x)\|_{\mathcal{C}^\beta} \\
&= \exp\{L(\nu_{N+1}, \rho_{N+1}; t-\nu_{N+1}) + (2-\kappa)(t-\nu_{N+1})\} \\
&\quad \times \exp\{-(2-\kappa)(t-\nu_{N+1})\} \\
&\quad \times \|X(\nu_{N+1};y) - X(\nu_{N+1};x)\|_{\mathcal{C}^\beta} \\
&\overset{(5.20),\omega\in\mathcal{S}(x)}{\lesssim} \exp\{-(2-\kappa)t\}\|y-x\|_{\mathcal{C}^\beta}.
\end{aligned}$$

By Proposition 5.13 there exist $a_0 > 0$ and $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \leq \varepsilon_0$

$$\inf_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0} \mathbb{P}(\mathcal{S}(x)) \geq 1 - \mathrm{e}^{-a_0/\varepsilon}$$

which completes the proof. □

We are now ready to prove Theorem 5.3 and Corollary 5.4.

Proof of Theorem 5.3. This is a consequence of (5.13), Proposition 5.19 and Theorem 5.15. Let $\delta_1, \delta_2 > 0$ sufficiently small such that $\delta_1 + \delta_2 < \delta_0$ and assume that $\tau_1(x) \geq 1$. By the definition of $\tau_1(x)$

$$\|X(1;x) - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \|v(1;x) - (\pm 1)\|_{\mathcal{C}^\beta} + \|\varepsilon^{\frac{1}{2}} \mathbf{↿}(1)\|_{\mathcal{C}^{-\alpha_0}} < \delta_1 + \delta_2 < \delta_0.$$

If we also choose $\delta_0' < \delta_0$ by (5.13) we have that for every $\|y-x\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0'$

$$\|X(1;y) - X(1;x)\|_{\mathcal{C}^\beta} \lesssim \|y-x\|_{\mathcal{C}^{-\alpha_0}}.$$

The probability of the event $\{\tau_1(x) \geq 1\}$ can be estimated from below by Proposition 5.19 uniformly in $\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0'$. Combining with Theorem 5.15 completes the proof. □

Proof of Corollary 5.4. We only prove the case where initial conditions are close to the minimiser 1. We fix $\delta_0', \delta_1' > 0$ such that $2\delta_0' < \delta_0$ and $\delta_0' + \delta_1' < \delta_1$. By Proposition 5.18 if we chose δ_2 sufficiently small then

$$\sup_{t\leq 1} t^{(n-1)\alpha'} \|\curlyvee^{n}(t)\|_{\mathcal{C}^{-\alpha}} \leq \delta_2 \Rightarrow \sup_{t\leq 1} t^{\gamma} \|v(t;y) - 1\|_{\mathcal{C}^{\beta}} \leq \delta_1$$

uniformly for $\|y-1\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0'$. This together with (5.13) implies that for every $x, y \in B_{\mathcal{C}^{-\alpha_0}}(1;\delta_0')$

$$\|X(1;y) - X(1;x)\|_{\mathcal{C}^{\beta}} \lesssim \|y-x\|_{\mathcal{C}^{-\alpha_0}} \lesssim \delta_0'.$$

Let

$$\omega \in \mathcal{S} := \left\{ \sup_{\|y-1\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0'} \frac{\|X(t;y) - X(t;1)\|_{\mathcal{C}^{\beta}}}{\|y-1\|_{\mathcal{C}^{-\alpha_0}}} \leq C\mathrm{e}^{-(2-\kappa)t},\ \forall t \geq 1 \right\},$$

$t \geq 1$ and $y \in B_{\mathcal{C}^{-\alpha_0}}(-1;\delta_0')$. Then

$$\begin{aligned}
&\sup_{s\leq t\leq T} (t-s)^{\gamma} \|v_s(t;X(s;1)) - (\pm 1)\|_{\mathcal{C}^{\beta}} \leq \delta_1' \\
&\quad \Rightarrow \sup_{s\leq t\leq T} (t-s)^{\gamma} \|v_s(t;X(s;y)) - (\pm 1)\|_{\mathcal{C}^{\beta}} \leq \delta_1 \text{ for } T, s \geq 1. \\
&\|X(t;1) - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0' \Rightarrow \|X(t;y) - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0.
\end{aligned}$$

This implies that if we consider the process $X(t;y)$ for $t \geq 1$, the times $\nu_i(X(1;y))$ and $\rho_i(X(1;y))$ of Definition 5.7 for δ_0, δ_1 and δ_2 can be replaced by the times $\nu_i(X(1;1))$ and $\rho_i(X(1;1))$ for δ_0', δ_1' and the same δ_2. Hence the corresponding random walk $S_N(X(1;y))$ in Definition 5.11 can be replaced by $S_N(X(1;1))$.

We can now repeat the proof of Theorem 5.15 for the difference $X(t;y) - X(t;x)$, $t \geq 1$, step by step, replacing the event in (5.19) by

$$\mathcal{S} \cap \left\{ \sup_{t\leq 1} t^{(n-1)\alpha'} \|\curlyvee^{n}(t)\|_{\mathcal{C}^{-\alpha}} \leq \delta_2,\ S_N(X(1;1)) \geq 0 \text{ for every } N \geq 1 \right\}. \tag{5.22}$$

This allows us to prove that

$$\|X(t;y) - X(t;x)\|_{\mathcal{C}^{\beta}} \leq C\mathrm{e}^{-(2-\kappa)(t-1)} \|X(1;y) - X(1;x)\|_{\mathcal{C}^{\beta}} \leq C\mathrm{e}^{-(2-\kappa)t} \|y-x\|_{\mathcal{C}^{-\alpha_0}}$$

uniformly in $y, x \in B_{\mathcal{C}^{-\alpha_0}}(1;\delta_0')$.

To estimate the event in (5.22) we use Theorem 5.3 and Propositions 5.17 and 5.13. This completes the proof. $\square$

5.2.2 Pathwise Estimates on the Difference of Two Profiles

In this section we prove Propositions 5.8 and 5.10. Our analysis here is pathwise and uses no probabilistic tools.

5.2.2.1 Proof of Proposition 5.8

Proof of Proposition 5.8. We only prove (5.13). To prove (5.12) we follow the same strategy as below. However in this case we do not need to encounter the blow-up of $\|Y(t)\|_{\mathcal{C}^\beta}$ close to 0 and hence we omit the proof since it poses no extra difficulties.

Let $Y(t) = X(t;y) - X(t;x)$ and notice that from (5.6) we get

$$(\partial_t - \Delta)Y = -\big(v(\cdot;y)^3 - v(\cdot;x)^3\big) + Y - 3(v(\cdot;y) + v(\cdot;x))\varepsilon^{\frac{1}{2}}\mathord{\uparrow}Y - 3\varepsilon\mathbf{V}Y.$$

We use the identity $v(\cdot;y) = v(\cdot;x) + Y$ to rewrite this equation in the form

$$\begin{aligned}&(\partial_t - (\Delta - 2))\,Y\\ &\quad = -3\left(v(\cdot;x)^2 - 1\right)Y + \mathtt{Error}(v(\cdot;x);Y) - 3(Y + 2v(\cdot;x))\varepsilon^{\frac{1}{2}}\mathord{\uparrow}Y - 3\varepsilon\mathbf{V}Y\end{aligned}$$

where $\mathtt{Error}(v(\cdot;x);Y) = -Y^3 - 3v(\cdot;x)Y^2$ collects all the terms which are higher order in Y. Then

$$\begin{aligned}Y(t) = \mathrm{e}^{-2t}\mathrm{e}^{\Delta t}Y(0) + \int_0^t \mathrm{e}^{-2(t-s)}\mathrm{e}^{\Delta(t-s)}\Big[&-3\left(v(s;x)^2 - 1\right)Y(s)\\ &+ \mathtt{Error}(v(s;x);Y(s)) - 3(Y(s) + 2v(s;x))\varepsilon^{\frac{1}{2}}\mathord{\uparrow}(s)Y(s)\\ &- 3\varepsilon\mathbf{V}(s)Y(s)\Big]\,\mathrm{d}s. \end{aligned} \tag{5.23}$$

We set $\tilde{\kappa} = \sup_{t\le\tau_1(x)}(t\wedge 1)^{2\gamma}\|-3\left(v(t;x)^2 - 1\right)\|_{\mathcal{C}^\beta}$. Let $\iota = \inf\{t > 0 : (t\wedge 1)^\gamma\|Y(t)\|_{\mathcal{C}^\beta} > \zeta\}$ for $1 \ge \zeta > \delta_0$ and notice that for $t \le \tau_1(x)\wedge\iota$ using (5.23) we get

$$\begin{aligned}\|Y(t)\|_{\mathcal{C}^\beta} \overset{\text{(A.7),(A.10),(A.11)}}{\le}\; & \mathrm{e}^{-2t}C(t\wedge 1)^{-\frac{\alpha_0+\beta}{2}}\|Y(0)\|_{\mathcal{C}^{-\alpha_0}}\\ &+ \tilde{\kappa}\int_0^t \mathrm{e}^{-2(t-s)}(s\wedge 1)^{-2\gamma}\|Y(s)\|_{\mathcal{C}^\beta}\,\mathrm{d}s\\ &+ \zeta\,C_1\int_0^t \mathrm{e}^{-2(t-s)}(s\wedge 1)^{-2\gamma}\|Y(s)\|_{\mathcal{C}^\beta}\,\mathrm{d}s\\ &+ \delta_2\,C_2\int_0^t \mathrm{e}^{-2(t-s)}(t-s)^{-\frac{\alpha+\beta}{2}}(s\wedge 1)^{-\gamma}\|Y(s)\|_{\mathcal{C}^\beta}\,\mathrm{d}s\\ &+ \delta_2\,C_3\int_0^t \mathrm{e}^{-2(t-s)}(t-s)^{-\frac{\alpha+\beta}{2}}(s\wedge 1)^{-\alpha'}\|Y(s)\|_{\mathcal{C}^\beta}\,\mathrm{d}s\end{aligned}$$

were we also use that for $s \le t$

$$\|\texttt{Error}(v(s;x);Y(s))\|_{\mathcal{C}^\beta} \lesssim \zeta s^{-2\gamma}\|Y(s)\|_{\mathcal{C}^\beta}.$$

Choosing $\zeta \le \tilde{\kappa}/C_1$ and $\delta_2 \le \tilde{\kappa}/C_2 \vee C_3$ we have

$$\begin{aligned}\|Y(t)\|_{\mathcal{C}^\beta} \le{}& \mathrm{e}^{-2t}C(t\wedge 1)^{-\frac{\alpha+\beta}{2}}\|Y(0)\|_{\mathcal{C}^{-\alpha_0}} \\ &+ \tilde{\kappa}\int_0^t \mathrm{e}^{-2(t-s)}(t-s)^{-\frac{\alpha+\beta}{2}}(s\wedge 1)^{-2\gamma}\|Y(s)\|_{\mathcal{C}^\beta}\,\mathrm{d}s.\end{aligned}$$

Then for $t \le \tau_1(x)\wedge\iota$ by Lemma F.1 on $f(t) = (t\wedge 1)^\gamma\|Y(t)\|_{\mathcal{C}^\beta}$ there exist $c > 0$ such that

$$(t\wedge 1)^\gamma\|Y(t)\|_{\mathcal{C}^\beta} \le C\exp\left\{-2t + c\tilde{\kappa}^{\overline{\frac{1}{1-\frac{\alpha+\beta}{2}-3\gamma}}}t + M\right\}\|Y(0)\|_{\mathcal{C}^{-\alpha_0}}.$$

We now fix $\delta_1 > 0$ such that $c\tilde{\kappa}^{\overline{\frac{1}{1-\frac{\alpha+\beta}{2}-3\gamma}}} \le \frac{\kappa}{2}$. This implies that for $t \le \tau_1(x)\wedge\iota$

$$(t\wedge 1)^\gamma\|Y(t)\|_{\mathcal{C}^\beta} \le C\exp\left\{-\left(2-\frac{\kappa}{2}\right)t\right\}\|Y(0)\|_{\mathcal{C}^{-\alpha_0}}.$$

Finally choosing δ_0 sufficiently small we furthermore notice that $\tau_1(x)\wedge\iota = \tau_1(x)$ which completes the proof of (5.13). $\square$

5.2.2.2 Proof of Proposition 5.10

To prove Proposition 5.10 we first need the following lemma which upgrades the a priori estimates in Proposition 3.10. Here and below we let $S(t) = \mathrm{e}^{\Delta t}$.

Lemma 5.16. *There exist* $\alpha, \gamma', C > 0$ *and* $p_0 \ge 1$ *such that if*

$$\sup_{t\le 1} t^{(n-1)\alpha'}\|\varepsilon^{\frac{n}{2}}\triangledown^{n}(t)\|_{\mathcal{C}^{-\alpha}} \le L^n$$

then

$$\sup_{x\in\mathcal{C}^{-\alpha_0}}\sup_{t\le 1} t^{\gamma'}\|v(t;x)\|_{\mathcal{C}^\beta} \le C(1\vee L)^{p_0}.$$

Proof. Throughout this proof we simply write $v(t)$ to denote $v(t;x)$. By Proposition 3.10 we have that for every $p \ge 2$ even

$$\sup_{x\in\mathcal{C}^{-\alpha_0}}\sup_{t\le 1} t^{\frac{1}{2}}\|v(t)\|_{L^p} \le C\left(1\vee\sup_{t\le 1} t^{(n-1)\alpha' p_n}\|\varepsilon^{\frac{n}{2}}\triangledown^{n}(t)\|^{p_n}_{\mathcal{C}^{-\alpha}}\right) \tag{5.24}$$

for some exponents $p_n \geq 1$. Combining (3.16) and (3.25) and integrating from s to t we obtain

$$\|v(t)\|_{L^2}^2 - \|v(s)\|_{L^2}^2 + \int_s^t \|\nabla v(r)\|_{L^2}^2 \,\mathrm{d}r \leq C \int_s^t \Big(1 + \sum_{n \leq 3} \|\varepsilon^{\frac{n}{2}} \Psi^{:n:}(r)\|_{\mathcal{C}^{-\alpha}}^{p_n}\Big) \,\mathrm{d}r$$

which implies that

$$\int_s^t \|\nabla v(r)\|_{L^2}^2 \,\mathrm{d}r \leq C \int_s^t \Big(1 + \sum_{n \leq 3} \|\varepsilon^{\frac{n}{2}} \Psi^{:n:}(r)\|_{\mathcal{C}^{-\alpha}}^{p_n}\Big) \,\mathrm{d}r + \|v(s)\|_{L^2}^2. \tag{5.25}$$

Using the mild form of (5.6) we have for $1 \geq t > s > 0$

$$\|v(t)\|_{\mathcal{C}^\beta} \lesssim \sum_{i=1}^{7} I_i \tag{5.26}$$

where

$$\begin{aligned}
&I_1 := \|S(t-s)v(s)\|_{\mathcal{C}^\beta},\ I_2 := \int_s^t \|S(t-r)v(r)^3\|_{\mathcal{C}^\beta} \,\mathrm{d}r, \\
&I_3 := \int_s^t \|S(t-r)\Big(v(r)^2 \varepsilon^{\frac{1}{2}} \Psi(r)\Big)\|_{\mathcal{C}^\beta} \,\mathrm{d}r, \\
&I_4 := \int_s^t \|S(t-r)\left(v(r)\varepsilon \Psi^{:2:}(r)\right)\|_{\mathcal{C}^\beta} \,\mathrm{d}r,\ I_5 = \int_s^t \|S(t-r)\varepsilon^{\frac{3}{2}} \Psi^{:3:}(r)\|_{\mathcal{C}^\beta} \,\mathrm{d}r, \\
&I_6 := \int_s^t \|S(t-r)\varepsilon^{\frac{1}{2}} \Psi(r)\|_{\mathcal{C}^\beta} \,\mathrm{d}r,\ I_7 := \int_s^t \|S(t-r)v(r)\|_{\mathcal{C}^\beta} \,\mathrm{d}r.
\end{aligned}$$

To estimate $\|v(t)\|_{\mathcal{C}^\beta}$ we use the L^p bound (5.24), the energy inequality (5.25) and the embedding $\mathcal{B}_{2,\infty}^1$ to bound the terms appearing on the right hand side of the last inequality as shown below.

We treat each term in (5.26) separately. Below p may change from term to term and α, λ can be taken arbitrarily small. We write p_1 and p_2 for conjugate exponents of p, i.e. $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. We also denote by $(1 \vee L)^{p_0}$ a polynomial of degree $p_0 \geq 1$ in the variable $1 \vee L$ where the value of p_0 may change from line to line.

<u>Term I_1</u>:

$$I_1 \overset{\text{(A.6),(A.7)}}{\lesssim} (t-s)^{-\frac{\beta + \frac{2}{p}}{2}} \|v(s)\|_{L^p} \overset{\text{(5.24)}}{\lesssim} (t-s)^{-\frac{\beta + \frac{2}{p}}{2}} s^{-\frac{1}{2}} (1 \vee L)^{p_0}.$$

<u>Term I_2</u>:

$$\begin{aligned}
I_2 &\overset{\text{(A.6),(A.7)}}{\lesssim} \int_s^t (t-r)^{-\frac{\beta + \frac{2}{p}}{2}} \|v(r)^3\|_{L^p} \,\mathrm{d}r \overset{\text{(5.24)}}{\lesssim} (1 \vee L)^{p_0} \int_s^t (t-r)^{-\frac{\beta + \frac{2}{p}}{2}} r^{-\frac{3}{2}} \,\mathrm{d}r \\
&\lesssim (1 \vee L)^{p_0} s^{-\frac{3}{2}} \int_s^t (t-r)^{-\frac{\beta + \frac{2}{p}}{2}} \,\mathrm{d}r.
\end{aligned}$$

Term I_3:

$$
\begin{aligned}
I_3 &\overset{\text{(A.6),(A.7),(A.11)},\lambda>0}{\lesssim} \int_s^t (t-r)^{-\frac{2\alpha+\lambda+\frac{2}{p}}{2}} \|v(r)^2\|_{\mathcal{B}^{\alpha+\lambda}_{p,\infty}} \|\varepsilon^{\frac12}\mathbf{I}(r)\|_{\mathcal{C}^{-\alpha}}\,\mathrm{d}r \\
&\overset{\text{(A.14)}}{\lesssim} \int_s^t (t-r)^{-\frac{2\alpha+\lambda+\frac{2}{p}}{2}} \|v(r)\|_{L^{p_1}} \|v(r)\|_{\mathcal{B}^{\alpha+\lambda}_{p_2,\infty}} \|\varepsilon^{\frac12}\mathbf{I}(r)\|_{\mathcal{C}^{-\alpha}}\,\mathrm{d}r \\
&\overset{\text{(A.6),(5.24)}}{\lesssim} \int_s^t (t-r)^{-\frac{2\alpha+\lambda+\frac{2}{p}}{2}} r^{-\frac12} \|v(r)\|_{\mathcal{B}^{\alpha+\lambda+1-\frac{2}{p_2}}_{2,\infty}} \|\varepsilon^{\frac12}\mathbf{I}(r)\|_{\mathcal{C}^{-\alpha}}\,\mathrm{d}r \\
&\overset{\text{(5.24)},\frac{2}{p_2}=\alpha+\lambda}{\lesssim} (1\vee L)^{p_0} s^{-\frac12} \int_s^t (t-r)^{-\frac{2\alpha+\lambda+\frac{2}{p}}{2}} \|v(r)\|_{\mathcal{B}^1_{2,\infty}}\,\mathrm{d}r \\
&\overset{\text{Cauchy–Schwarz}}{\lesssim} (1\vee L)^{p_0} s^{-\frac12} \left(\int_s^t (t-r)^{-\left(2\alpha+\lambda+\frac{2}{p}\right)}\,\mathrm{d}r\right)^{\frac12} \\
&\quad\times \left(\int_s^t \|v(r)\|^2_{\mathcal{B}^1_{2,\infty}}\,\mathrm{d}r\right)^{\frac12}.
\end{aligned}
$$

Term I_4:

$$
\begin{aligned}
I_4 &\overset{\text{(A.6),(A.7),(A.11)},\lambda>0}{\lesssim} \int_s^t (t-r)^{-\frac{2\alpha+\lambda+\frac{2}{p}}{2}} \|v(r)\|_{\mathcal{B}^{\alpha+\lambda}_{p,\infty}} \|\varepsilon\mathbf{V}(r)\|_{\mathcal{C}^{-\alpha}}\,\mathrm{d}r \\
&\overset{\text{(A.6)}}{\lesssim} \int_s^t (t-r)^{-\frac{2\alpha+\lambda+\frac{2}{p}}{2}} \|v(r)\|_{\mathcal{B}^{\alpha+\lambda+1-\frac{2}{p}}_{2,\infty}} \|\varepsilon\mathbf{V}(r)\|_{\mathcal{C}^{-\alpha}}\,\mathrm{d}r \\
&\overset{\frac{2}{p}=\alpha+\lambda}{\lesssim} (1\vee L)^{p_0} s^{-\alpha'} \int_s^t (t-r)^{-\frac{2\alpha+\lambda+\frac{2}{p}}{2}} \|v(r)\|_{\mathcal{B}^1_{2,\infty}}\,\mathrm{d}r \\
&\overset{\text{Cauchy–Schwarz}}{\lesssim} (1\vee L)^{p_0} s^{-\alpha'} \left(\int_s^t (t-r)^{-\left(2\alpha+\lambda+\frac{2}{p}\right)}\,\mathrm{d}r\right)^{\frac12} \\
&\quad\times \left(\int_s^t \|v(r)\|^2_{\mathcal{B}^1_{2,\infty}}\,\mathrm{d}r\right)^{\frac12}.
\end{aligned}
$$

Term I_5:

$$
\begin{aligned}
I_5 &\overset{\text{(A.7)}}{\lesssim} \int_s^t (t-r)^{-\frac{\alpha+\beta}{2}} \|\varepsilon^{\frac32}\mathbf{\Psi}(r)\|_{\mathcal{C}^{-\alpha}}\,\mathrm{d}r \lesssim (1\vee L)^{p_0} \int_s^t (t-r)^{-\frac{\alpha+\beta}{2}} r^{-2\alpha'}\,\mathrm{d}r \\
&\lesssim (1\vee L)^{p_0} s^{-2\alpha'} \int_s^t (t-r)^{-\frac{\alpha+\beta}{2}}\,\mathrm{d}r.
\end{aligned}
$$

Term I_6:

$$
I_6 \overset{\text{(A.7)}}{\lesssim} \int_s^t (t-r)^{-\frac{\alpha+\beta}{2}} \|\varepsilon^{\frac12}\mathbf{I}(r)\|_{\mathcal{C}^{-\alpha}}\,\mathrm{d}r \lesssim (1\vee L)^{p_0} \int_s^t (t-r)^{-\frac{\alpha+\beta}{2}}\,\mathrm{d}r.
$$

Term I_7:

$$I_7 \overset{\text{(A.6),(A.7)}}{\lesssim} \int_s^t (t-r)^{-\frac{\beta+\frac{2}{p}}{2}} \|v(r)\|_{L^p} \,\mathrm{d}r \overset{\text{(5.24)}}{\lesssim} (1\vee L)^{p_0} \int_s^t (t-r)^{-\frac{\beta+\frac{2}{p}}{2}} r^{-\frac{1}{2}} \,\mathrm{d}r$$
$$\lesssim (1\vee L)^{p_0} s^{-\frac{1}{2}} \int_s^t (t-r)^{-\frac{\beta+\frac{2}{p}}{2}} \,\mathrm{d}r.$$

Using Proposition A.11, (5.24) and (5.25) we notice that

$$\left(\int_s^t \|v(r)\|^2_{\mathcal{B}^1_{2,\infty}} \,\mathrm{d}r\right)^{\frac{1}{2}} \lesssim \left(\int_s^t \|\nabla v(r)\|^2_{L^2} \,\mathrm{d}r\right)^{\frac{1}{2}} + \left(\int_s^t \|v(r)\|^2_{L^2} \,\mathrm{d}r\right)^{\frac{1}{2}}$$
$$\lesssim (1\vee L)^{p_0} s^{-\frac{1}{2}}.$$

Combining the above and choosing $s = t/2$ we find $\gamma' > 0$ such that

$$t^{\gamma'} \|v(t)\|_{\mathcal{C}^\beta} \lesssim (1\vee L)^{p_0}$$

which completes the proof. □

Proof of Proposition 5.10. We denote by $(1\vee L)^{p_0}$ a polynomial of degree $p_0 \geq 1$ in the variable $1\vee L$ where the value of p_0 may change from line to line.

For $k \geq 0$ recall that $t_k = \nu + k$ and $s_k = t_k + \frac{1}{2}$. As before, we write $Y(t) = X(t;y) - X(t;x)$.

Let $t \in (t_k, s_k]$, $k \geq 1$. We restart the stochastic terms at time s_{k-1} and write $Y(t) = v_{s_{k-1}}(t;\tilde{y}) - v_{s_{k-1}}(t;\tilde{x})$ where for simplicity $\tilde{y} = X(s_{k-1};y)$ and $\tilde{x} = X(s_{k-1};x)$. Together with (5.10), this implies that

$$(\partial_t - \Delta)Y = -\left(v_{s_{k-1}}(\cdot;\tilde{y})^3 - v_{s_{k-1}}(\cdot;\tilde{x})^3\right) + Y$$
$$- 3(v_{s_{k-1}}(\cdot;\tilde{y}) + v_{s_{k-1}}(\cdot;\tilde{x}))\varepsilon^{\frac{1}{2}} \mathbf{!}_{s_{k-1}} Y - 3\varepsilon \mathbf{V}_{s_{k-1}} Y.$$

Using the mild form of the above equation, now starting at $t_k = s_{k-1} + \frac{1}{2}$, we get

$$\|Y(t)\|_{\mathcal{C}^\beta} \overset{\text{(A.7),(A.10),(A.11)}}{\lesssim} \|Y(t_k)\|_{\mathcal{C}^\beta} + \int_{t_k}^t \|v_{s_{k-1}}(r;\tilde{y})^3 - v_{s_{k-1}}(r;\tilde{x})^3\|_{\mathcal{C}^\beta} \,\mathrm{d}r$$
$$+ \int_{t_k}^t (t-r)^{-\frac{\alpha+\beta}{2}} \|v_{s_{k-1}}(r;\tilde{y})^2 - v_{s_{k-1}}(r;\tilde{x})^2\|_{\mathcal{C}^\beta} \|\varepsilon^{\frac{1}{2}} \mathbf{!}_{s_{k-1}}(r)\|_{\mathcal{C}^{-\alpha}} \,\mathrm{d}r$$
$$+ \int_{t_k}^t (t-r)^{-\frac{\alpha+\beta}{2}} \|Y(r)\|_{\mathcal{C}^\beta} \|\varepsilon \mathbf{V}_{s_{k-1}}(r)\|_{\mathcal{C}^{-\alpha}} \,\mathrm{d}r + \int_{t_k}^t \|Y(r)\|_{\mathcal{C}^\beta} \,\mathrm{d}r.$$

By Lemma 5.16 there exist $\gamma' > 0$ such that

$$\sup_{x\in\mathcal{C}^{-\alpha_0}} \sup_{t\in[s_{k-1},s_k]} (t - s_{k-1})^{\gamma'} \|v_{s_{k-1}}(t;x)\|_{\mathcal{C}^\beta} \lesssim \left(1 \vee L_k\left(\nu + \frac{1}{2}, \rho\right)\right)^{p_0}.$$

Combining the above we get

$$\|Y(t)\|_{\mathcal{C}^\beta} \lesssim \|Y(t_k)\|_{\mathcal{C}^\beta} + \left(1 \vee L_k\left(\nu + \frac{1}{2}, \rho\right)\right)^{p_0} \int_{t_k}^{t} (t-r)^{-\frac{\alpha+\beta}{2}} \|Y(r)\|_{\mathcal{C}^\beta}\, \mathrm{d}r.$$

By Lemma F.1 there exists $c_0 > 0$ such that

$$\|Y(t)\|_{\mathcal{C}^\beta} \lesssim \exp\left\{c_0 \left(1 \vee L_k\left(\nu + \frac{1}{2}, \rho\right)\right)^{p_0} (t-s)\right\} \|Y(t_k)\|_{\mathcal{C}^\beta}. \tag{5.27}$$

Following the same strategy we prove that for $t \in [s_k, t_{k+1}]$, $k \geq 1$,

$$\|Y(t)\|_{\mathcal{C}^\beta} \lesssim \exp\left\{c_0 \left(1 \vee L_{k+1}(\nu, \rho)\right)^{p_0} (t-s)\right\} \|Y(s_k)\|_{\mathcal{C}^\beta}. \tag{5.28}$$

Finally, we also need a bound for $t \in [t_0, t_1]$. To obtain an estimate which does not depend on any information before time t_0 we use local solution theory. By Theorem 3.6 there exists $t_* \in (t_0, t_1)$ such that

$$\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}} \leq R} \sup_{r \in [t_0, t_*]} (r-t_0)^\gamma \|v_{t_0}(r;x)\|_{\mathcal{C}^\beta} \leq 1$$

and furthermore we can take

$$t_* = \left(\frac{1}{C(R \vee L_1(\nu,\rho))}\right)^{p_0}.$$

By Lemma 5.16 we also have that

$$\sup_{x \in \mathcal{C}^{-\alpha_0}} \sup_{r \in (t_0, t_1]} (r-t_0)^{\gamma'} \|v_{t_0}(r;x)\|_{\mathcal{C}^\beta} \lesssim (1 \vee L_1(\nu,\rho))^{p_0}.$$

Combining these two bounds we get

$$\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}} \leq R} \sup_{r \in [t_0, t_1]} (r-t_0)^\gamma \|v_{t_0}(r;x)\|_{\mathcal{C}^\beta} \lesssim (1 \vee L_1(\nu,\rho))^{p_0} \tag{5.29}$$

were the implicit constant depends on R. Note that $\gamma < \frac{1}{3}$ whereas γ' is much larger. We write $Y(t) = v_{t_0}(t;y) - v_{t_0}(t;x)$ and use the mild form starting at t_0. We then use (5.29) to bound $\|v_{t_0}(t;\cdot)\|_{\mathcal{C}^\beta}$ on $[t_0, t_1]$ which implies the estimate

$$\|Y(t)\|_{\mathcal{C}^\beta} \lesssim \|Y(t_0)\|_{\mathcal{C}^\beta} + (1 \vee L_1(\nu,\rho))^{p_0} \int_{t_0}^{t} (t-r)^{-\frac{\alpha+\beta}{2}} (r-t_0)^{-2\gamma} \|Y(r)\|_{\mathcal{C}^\beta}\, \mathrm{d}r.$$

The extra term $(r-t_0)^{-2\gamma}$ in the last inequality appears because of the blow-up of $v_{t_0}(t;\cdot)$ and $\triangledown_{t_0}(t)$ for t close to t_0. By Lemma F.1 we obtain that

$$\|Y(t)\|_{\mathcal{C}^\beta} \lesssim \exp\left\{c_0 \left(1 \vee L_1(\nu, \rho)\right)^{p_0} (t-s)\right\} \|Y(s)\|_{\mathcal{C}^\beta}. \tag{5.30}$$

For arbitrary $t \in [\nu, \rho]$ we glue together (5.27), (5.28) and (5.30) to get

$$\|Y(t)\|_{\mathcal{C}^\beta} \lesssim \exp\left\{ \frac{c_0}{2} \sum_{k=1}^{\lfloor t-\nu \rfloor} \sum_{l=0,\frac{1}{2}} (1 \vee L_k(\nu + l, \rho))^{p_0} + L_0(t-\nu) \right\} \|Y(\nu)\|_{\mathcal{C}^\beta}$$

for some $L_0 > 0$ which collects the implicit constants in the inequalities. $\square$

5.2.3 Random Walk Estimates

In this section we prove Proposition 5.13 based mainly on probabilistic arguments. In Sections 5.2.3.1 and 5.2.3.2 we provide estimates on $\frac{\kappa}{2}\tau_i(x)$ and $L(\nu_i(x), \rho_i(x); \sigma_i(x)) + (2-\kappa)\sigma_i(x) + M_0$ from Definition 5.11. In Section 5.2.3.3 we use these estimates to prove Proposition 5.13.

5.2.3.1 Estimates on the Exit Times

Proposition 5.17. *Let $\delta > 0$ and $\tau_{\text{tree}} = \inf\{t > 0 : (t \wedge 1)^{(n-1)\alpha'} \|\varepsilon^{\frac{n}{2}} \overset{n}{\curlyvee}(t)\|_{\mathcal{C}^{-\alpha}} \geq \delta^n\}$. Then there exist $a_0 > 0$ and $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \leq \varepsilon_0$*

$$\mathbb{P}\left(\tau_{\text{tree}} \leq \mathrm{e}^{3a_0/\varepsilon}\right) \leq \mathrm{e}^{-3a_0/\varepsilon}.$$

Proof. First notice that for $N \geq 1$

$$\begin{aligned} \mathbb{P}(\tau_{\text{tree}} \leq N) &\leq \sum_{k=0}^{N-1} \mathbb{P}(\tau_{\text{tree}} \in (k, k+1)) \\ &\leq \sum_{k=0}^{N-1} \mathbb{P}\left(\sup_{t \in (k,k+1]} (t \wedge 1)^{(n-1)\alpha'} \|\varepsilon^{\frac{n}{2}} \overset{n}{\curlyvee}(t)\|_{\mathcal{C}^{-\alpha}} \geq \delta^n \right). \end{aligned}$$

By Proposition H.1 and the exponential Chebyshev inequality there exists $a_0 > 0$ such that for every $k \geq 0$

$$\mathbb{P}\left(\sup_{t \in (k,k+1]} (t \wedge 1)^{(n-1)\alpha'} \|\varepsilon^{\frac{n}{2}} \overset{n}{\curlyvee}(t)\|_{\mathcal{C}^{-\alpha}} \geq \delta^n \right) \leq \mathrm{e}^{-6a_0/\varepsilon}.$$

Hence

$$\mathbb{P}(\tau_{\text{tree}} \leq N) \leq N \mathrm{e}^{-6a_0/\varepsilon}$$

and choosing $N = \mathrm{e}^{3a_0/\varepsilon}$ completes the proof. $\square$

Proposition 5.18. *For $\delta_1 > 0$ sufficiently small there exist $\delta_0, \delta_2 > 0$ such that if*

$$\sup_{t \le T} (t \wedge 1)^{(n-1)\alpha'} \| \varepsilon^{\frac{n}{2}} \overset{n}{\vee}(t) \|_{\mathcal{C}^{-\alpha}} < \delta_2^n \tag{5.31}$$

then for every $\|x - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0$

$$\sup_{t \le T} (t \wedge 1)^{\gamma} \| v(t;x) - (\pm 1) \|_{\mathcal{C}^{\beta}} < \delta_1$$

and

$$\sup_{t \le T} \| X(t;x) - (\pm 1) \|_{\mathcal{C}^{-\alpha_0}} \le 2\delta_1 .$$

Proof. Let $u(t) = v(t;x) - (\pm 1)$. A Taylor expansion of $-v^3 + v$ around ± 1 implies that

$$(\partial_t - (\Delta - 2))u = \mathtt{Error}(u) - \left(3v^2 \varepsilon^{\frac{1}{2}} \uparrow + 3v\varepsilon \vee + \varepsilon^{\frac{3}{2}} \Psi \right) + 2\varepsilon^{\frac{1}{2}} \uparrow \tag{5.32}$$

where $\mathtt{Error}(u) = -u^3 \pm 3u^2$ and $\|\mathtt{Error}(u)\|_{\mathcal{C}^{\beta}} \lesssim \|u\|_{\mathcal{C}^{\beta}}^3 + \|u\|_{\mathcal{C}^{\beta}}^2$. Let $T > 0$ and $\iota = \inf\{t > 0 : (t \wedge 1)^{\gamma} \|u(t)\|_{\mathcal{C}^{\beta}} \ge \delta_1\}$ for some $\delta_1 > 0$ which we fix below. Using the mild form of (5.32) we get

$$\begin{aligned}
&(t \wedge 1)^{\gamma} \|u(t)\|_{\mathcal{C}^{\beta}} \\
&\quad \overset{(A.7),(A.10),(A.11)}{\lesssim} \mathrm{e}^{-2t} \|x - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} + \int_0^t \mathrm{e}^{-2(t-s)} \left(\|u(s)\|_{\mathcal{C}^{\beta}}^3 + \|u(s)\|_{\mathcal{C}^{\beta}}^2 \right) \mathrm{d}s \\
&\quad + \int_0^t \mathrm{e}^{-2(t-s)} (t-s)^{-\frac{\alpha+\beta}{2}} \Big(\|v(s)\|_{\mathcal{C}^{\beta}}^2 \|\varepsilon^{\frac{1}{2}} \uparrow(s)\|_{\mathcal{C}^{-\alpha}} + \|v(s)\|_{\mathcal{C}^{\beta}} \|\varepsilon \vee(s)\|_{\mathcal{C}^{-\alpha}} \\
&\qquad + \|\varepsilon^{\frac{3}{2}} \Psi(s)\|_{\mathcal{C}^{-\alpha}} + \|\varepsilon^{\frac{1}{2}} \uparrow(s)\|_{\mathcal{C}^{-\alpha}} \Big) \, \mathrm{d}s.
\end{aligned}$$

If we furthermore assume (5.31) for $t \le T \wedge \iota$ we obtain that

$$\begin{aligned}
&(t \wedge 1)^{\gamma} \|u(t)\|_{\mathcal{C}^{\beta}} \\
&\quad \lesssim \delta_0 \mathrm{e}^{-2t} + \delta_1^3 \int_0^t \mathrm{e}^{-2(t-s)} (s \wedge 1)^{-3\gamma} \, \mathrm{d}s + \delta_1^2 \int_0^t \mathrm{e}^{-2(t-s)} (s \wedge 1)^{-2\gamma} \, \mathrm{d}s \\
&\quad + \delta_2 \int_0^t \mathrm{e}^{-2(t-s)} (t-s)^{-\frac{\alpha+\beta}{2}} \Big((s \wedge 1)^{-2\gamma} + (s \wedge 1)^{-\gamma} (s \wedge 1)^{-\alpha'} \\
&\qquad + (s \wedge 1)^{-2\alpha'} + 1 \Big) \, \mathrm{d}s.
\end{aligned}$$

Then Lemma F.2 implies the bound

$$\sup_{t \le T \wedge \iota} (t \wedge 1)^{\gamma} \|u(t)\|_{\mathcal{C}^{\beta}} \lesssim \delta_0 + \delta_1^3 + \delta_1^2 + \delta_2 .$$

Choosing $\delta_0 < \frac{\delta_1}{4C}$, $\delta_1 < \frac{1}{4C}$ and $\delta_2 < \frac{\delta_1}{4C}$ this implies that

$$\sup_{t\le T\wedge\iota} (t\wedge 1)^\gamma \|u(t)\|_{\mathcal{C}^\beta} < \delta_1$$

which in turn implies that $\iota \le T$ and proves the first bound.

To prove the second bound we notice that for every $t \le T$

$$\|X(t;x) - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \le \|u(t)\|_{\mathcal{C}^{-\alpha_0}} + \|\mathbf{\uparrow}(t)\|_{\mathcal{C}^{-\alpha_0}} \le \|u(t)\|_{\mathcal{C}^{-\alpha_0}} + \delta_2.$$

Hence it suffices to prove that $\sup_{t\le T}\|u(t)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_1$. Using again the mild form of (5.32) we get

$$\begin{aligned}\|u(t)\|_{\mathcal{C}^{-\alpha_0}} &\overset{\text{(A.7),(A.1),(A.10),(A.11)}}{\lesssim} \mathrm{e}^{-2t}\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \\ &\quad + \int_0^t \mathrm{e}^{-2(t-s)}\left(\|u(s)\|_{\mathcal{C}^\beta}^3 + \|u(s)\|_{\mathcal{C}^\beta}^2\right)\mathrm{d}s \\ &\quad + \int_0^t \mathrm{e}^{-2(t-s)}\Big(\|v(s)\|_{\mathcal{C}^\beta}^2 \|\varepsilon^{\frac12}\mathbf{\uparrow}(s)\|_{\mathcal{C}^{-\alpha}} + \|v(s)\|_{\mathcal{C}^\beta}\|\varepsilon\mathbf{V}(s)\|_{\mathcal{C}^{-\alpha}} \\ &\qquad + \|\varepsilon^{\frac32}\mathbf{\Psi}(s)\|_{\mathcal{C}^{-\alpha}} + \|\varepsilon^{\frac12}\mathbf{\uparrow}(s)\|_{\mathcal{C}^{-\alpha}}\Big)\,\mathrm{d}s\end{aligned}$$

for every $t \le T$. Plugging in (5.31) and the bound $\sup_{t\le T}(t\wedge 1)^\gamma\|u(t)\|_{\mathcal{C}^\beta} \le \delta_1$ the last inequality implies

$$\begin{aligned}&\|u(t)\|_{\mathcal{C}^{-\alpha_0}} \\ &\quad\lesssim \delta_0\mathrm{e}^{-2t} + \delta_1^3\int_0^t \mathrm{e}^{-2(t-s)}(s\wedge 1)^{-3\gamma}\,\mathrm{d}s + \delta_1^2\int_0^t \mathrm{e}^{-2(t-s)}(s\wedge 1)^{-2\gamma}\,\mathrm{d}s \\ &\qquad + \delta_2\int_0^t \mathrm{e}^{-2(t-s)}\Big((s\wedge 1)^{-2\gamma} + (s\wedge 1)^{-\gamma}(s\wedge 1)^{-\alpha'} + (s\wedge 1)^{-2\alpha'} + 1\Big)\,\mathrm{d}s.\end{aligned}$$

Using again Lemma F.2 we obtain that $\sup_{t\le T}\|u(t)\|_{\mathcal{C}^{-\alpha_0}} < \delta_1$, which completes the proof. □

Proposition 5.19. *For every $\kappa > 0$ and $\delta_1 > 0$ sufficiently small there exist $a_0, \delta_0, \delta_2 > 0$ and $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \le \varepsilon_0$*

$$\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}}\le\delta_0} \mathbb{P}\left(\frac{\kappa}{2}\tau_1(x) \le \mathrm{e}^{2a_0/\varepsilon}\right) \le \mathrm{e}^{-3a_0/\varepsilon},$$

where $\tau_1(x)$ is given by (5.11).

Proof. We first notice that there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \le \varepsilon_0$

$$\mathbb{P}\left(\frac{\kappa}{2}\tau_1(x) \le \mathrm{e}^{2a_0/\varepsilon}\right) \le \mathbb{P}\left(\tau_1(x) \le \mathrm{e}^{3a_0/\varepsilon}\right).$$

The last probability can be estimated by Propositions 5.18 and 5.17 for $\delta = \delta_2$. □

5.2.3.2 Estimates on the Entry Times

In this section we use large deviation theory and in particular a lower bound of the form

$$\begin{aligned}\liminf_{\varepsilon\searrow 0}\log\varepsilon\inf_{x\in\aleph}\mathbb{P}(X(\cdot;x)\in\mathcal{A}(T;x))\\ \geq -\sup_{x\in\aleph}\inf_{\substack{f\in\mathcal{A}(T;x)\\ f(0)=x}}\left\{\underbrace{\frac{1}{4}\int_0^T\|(\partial_t-\Delta)f(t)+f(t)^3-f(t)\|_{L^2}^2\,\mathrm{d}t}_{=:I(f)}\right\}\end{aligned} \tag{5.33}$$

where $\aleph$ is a compact subset of $\mathcal{C}^{-\alpha}$ and $\mathcal{A}(T;x)\subset\{f:(0,T)\to\mathcal{C}^{-\alpha}\}$ is open. This bound is an immediate consequence of [HW15] and the remark that the solution map

$$\mathcal{C}^{-\alpha_0}\times\left(\mathcal{C}^{-\alpha}\right)^3\ni\left(x,\left\{\varepsilon^{\frac{n}{2}}\mathord{\vee}^{n}\right\}_{n\leq 3}\right)\mapsto X(\cdot;x)\in\mathcal{C}^{-\alpha}$$

is jointly continuous on compact time intervals. This estimate implies a “nice” lower bound for the probabilities $\mathbb{P}(X(\cdot;x)\in\mathcal{A}(T;x))$ if a suitable path $f\in\mathcal{A}(T;x)$ is chosen.

In the next proposition we use the lower bound (5.33) for suitable sets $\aleph$ and $\mathcal{A}(T;x)$ to estimate probabilities of the entry time of X in a neighbourhood of ±1. We construct a path $f(\cdot;x)$ and obtain bounds on $I(f(\cdot;x))$ uniformly in $x\in\aleph$.

Proposition 5.20. *Let $\delta_0>0$ and*

$$\sigma(x)=\inf\left\{t>0:\min_{x_*\in\{-1,1\}}\|X(t;x)-x_*\|_{\mathcal{C}^{-\alpha_0}}\leq\delta_0\right\}.$$

For every $R,b>0$ there exists $T_0>0$ such that

$$\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}}\leq R}\mathbb{P}(\sigma(x)\geq T_0)\leq 1-\mathrm{e}^{-b/\varepsilon}.$$

Proof. First notice that

$$\mathbb{P}(\sigma(x)\leq T_0)=\mathbb{P}(\underbrace{\|X(T_*;x)-(\pm1)\|_{\mathcal{C}^{-\alpha_0}}<\delta_0\text{ for some }T_*\leq T_0}_{=:\mathcal{A}(T_0;x)}).$$

By the large deviation estimate (5.33) it suffices to bound

$$\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}}\leq R}\inf_{\substack{f\in\mathcal{A}(T_0;x)\\ f(0)=x}}I(f(\cdot;x)).$$

We construct a suitable path $g \in \mathcal{A}(T_0; x)$ and we use the trivial inequality

$$\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}} \leq R} \inf_{\substack{f \in \mathcal{A}(T_0;x) \\ f(0)=x}} I(f(\cdot;x)) \leq \sup_{\|x\|_{\mathcal{C}^{-\alpha_0}} \leq R} I(g(\cdot;x)).$$

We now give the construction of g which involves 5 different steps. In Steps 1, 3 and 5, g follows the deterministic flow. The contribution of these steps to the energy functional I is zero. On Steps 2 and 3, g is constructed by linear interpolation. The contribution of these steps is estimated by Lemma 5.21. Below we write $X_{det}(\cdot; x)$ to denote the solution of (5.2) with initial condition x. We also pass through the space $\mathcal{B}^1_{2,2}$ to use convergence results for $X_{det}(\cdot; x)$ which hold in this topology (see Propositions G.1 and G.2).

Step 1 (Smoothness of initial condition via the deterministic flow):

Let $\tau_1 = 1$. For $t \in [0, \tau_1]$ we set $g(t; x) = X_{det}(t; x)$. By Proposition G.3 there exist $C \equiv C(r) > 0$ and $\lambda > 0$ such that

$$\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}} \leq R} \|X_{det}(1; x)\|_{\mathcal{C}^{2+\lambda}} \leq C.$$

Step 2 (Reach points that lead to a stationary solution):

By Step 1 $g(\tau_1; x) \in B_{\mathcal{C}^{2+\lambda}}(0; C)$ uniformly for $\|x\|_{\mathcal{C}^{-\alpha_0}} \leq R$. Let $\delta > 0$ to be fixed below. By compactness there exists $\{y_i\}_{1 \leq i \leq N}$ such that $B_{\mathcal{C}^{2+\lambda}}(0; C)$ is covered by $\cup_{1 \leq i \leq N} B_{\mathcal{B}^1_{2,2}}(y_i; \delta)$. Here we use that $\mathcal{C}^{2+\lambda}$ is compactly embedded in $\mathcal{B}^1_{2,2}$ (see Proposition A.4).

Without loss of generality we assume that $\{y_i\}_{1 \leq i \leq N}$ is such that $y_i \in \mathcal{C}^\infty$ and $X_{det}(t; y_i)$ converges to a stationary solution $-1, 0, 1$ in $\mathcal{B}^1_{2,2}$. Otherwise we choose $\{y_i^*\}_{1 \leq i \leq N} \in B_{\mathcal{B}^1_{2,2}}(y_i; \delta)$ such that $y_i^* \in \mathcal{C}^\infty$ and relabel them. This is possible because of Proposition G.1.

Let $\tau_2 = \tau_1 + \tau$, for $\tau > 0$ which we fix below. For $t \in [\tau_1, \tau_2]$ we set $g(t; x) = g(\tau_1; x) + \frac{t - \tau_1}{\tau_2 - \tau_1}(y_i - g(\tau_1; x))$, where y_i is such that $g(\tau_1; x) \in B_{\mathcal{B}^1_{2,2}}(y_i; \delta)$.

Step 3 (Follow the deterministic flow to reach a stationary solution):

Let T_i^* be such that $X_{det}(t; y_i) \in B_{\mathcal{B}^1_{2,2}}(x_*; \delta)$ for every $t \geq T_i^*$, where $x_* \in \{-1, 0, 1\}$ is the limit of $X_{det}(t; y_i)$ in $\mathcal{B}^1_{2,2}$, for $\{y_i\}_{1 \leq i \leq N}$ as in Step 2. Let $\tau_3 = \tau_2 + \max_{1 \leq i \leq N} T_i^* \vee 1$. For $t \in [\tau_2, \tau_3]$ we set $g(t; x) = X_{det}(t - \tau_2; y_i)$. If $X_{det}(\tau_3 - \tau_2; y_i) \in B_{\mathcal{B}^1_{2,2}}(\pm 1; \delta)$ we stop here. Otherwise $X_{det}(\tau_3 - \tau_2; y_i) \in B_{\mathcal{B}^1_{2,2}}(0; \delta) \cap B_{\mathcal{C}^{2+\lambda}}(0; C)$ (here we use again Proposition G.3 to ensure that $X_{det}(\tau_3 - \tau_2; y_i) \in B_{\mathcal{C}^{2+\lambda}}(0; C)$) and we proceed to Steps 4 and 5.

Step 4 (Move to a point nearby which leads to a stable solution):

We choose $y_0 \in B_{\mathcal{B}^1_{2,2}}(0;\delta)$ such that $y_0 \in \mathcal{C}^\infty$ and $X_{det}(t;y_0)$ converges to either 1 or -1 in $\mathcal{B}^1_{2,2}$. This is possible because of Proposition G.2.

Let $\tau_4 = \tau_3 + \tau$ for $\tau > 0$ as in Step 2 which we fix below. For $t \in [\tau_3, \tau_4]$ we set $g(t;x) = g(\tau_3;x) + \frac{t-\tau_3}{\tau_4-\tau_3}(y_0 - g(\tau_3;x))$.

Step 5 (Follow the deterministic flow again to finally reach a stable solution):

Let T_0^* be such that $X_{det}(t;y_0) \in B_{\mathcal{B}^1_{2,2}}(\pm 1;\delta)$ for every $t \geq T_0^*$, where y_0 is as in Step 4. Let $\tau_5 = \tau_4 + T_0^* \vee 1$. For $t \in [\tau_4, \tau_5]$ we set $g(t;x) = X_{det}(t-\tau_4;y_0)$.

For the path $g(\cdot;x)$ constructed above we see that after time $t \geq \tau_5$, $g(t;x) \in B_{\mathcal{B}^1_{2,2}}(\pm 1;\delta)$ for every $\|x\|{-}\mathcal{C}^{-\alpha_0} \leq R$. This implies that $\|g(t;x) - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} < C\delta$ since by (A.6), $\mathcal{B}^1_{2,2} \subset \mathcal{C}^{-\alpha_0}$. We now choose $\delta > 0$ such that $C\delta < \delta_0$ and let $T_0 = \tau_5 + 1$. Then $g \in \mathcal{A}(T_0;x)$.

To bound $I(g(\cdot;x))$ we split our time interval based on the construction of g i.e. $I_k = [\tau_{k-1}, \tau_k]$ for $k = 1, \ldots, 4$ and $I_5 = [\tau_5, T_0]$. We first notice that for $k = 1, 3, 5$

$$\frac{1}{4}\int_{I_k} \|(\partial_t - \Delta)g(t;x) + g(t;x)^3 - g(t;x)\|_{L^2}^2 \,\mathrm{d}t = 0$$

since on these intervals we follow the deterministic flow. For the remaining two intervals, i.e. $k = 2, 4$, we first notice that by construction

$$\|g(\tau_{k-1};x)\|_{\mathcal{C}^{2+\lambda}}, \|g(\tau_k;x)\|_{\mathcal{C}^{2+\lambda}} \leq C.$$

By (A.4), $\mathcal{C}^{2+\lambda} \subset \mathcal{B}^2_{\infty,2}$ for every $\lambda > 0$, hence we also have that

$$\|g(\tau_{k-1};x)\|_{\mathcal{B}^2_{\infty,2}}, \|g(\tau_k;x)\|_{\mathcal{B}^2_{\infty,2}} \leq C.$$

We can now choose τ in Steps 2 and 4 according to Lemma 5.21, which implies that

$$\frac{1}{4}\int_{I_k} \|(\partial_t - \Delta)g(t;x) + g(t;x)^3 - g(t;x)\|_{L^2}^2 \,\mathrm{d}t \leq C\delta.$$

Hence

$$\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}} \leq R} \frac{1}{4}\int_0^{T_0} \|(\partial_t - \Delta)g(t;x) + g(t;x)^3 - g(t;x))\|_{L^2}^2 \,\mathrm{d}t \leq C\delta.$$

For $b > 0$ we choose δ even smaller to ensure that $C\delta < b$. Finally, by (5.33) there exists $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \leq \varepsilon_0$

$$\inf_{\|x\|_{\mathcal{C}^{-\alpha_0}} \leq R} \mathbb{P}(\sigma(x) \leq T_0) \geq \mathrm{e}^{-b/\varepsilon}$$

which completes the proof. □

Lemma 5.21 ([FJL82, Lemma 9.2]). *Let* $f(t) = x + \frac{t}{\tau}(y-x)$ *such that*

$$\|x\|_{\mathcal{B}^2_{2,2}}, \|y\|_{\mathcal{B}^2_{2,2}} \leq R \text{ and } \|x-y\|_{L^2} \leq \delta.$$

There exist $\tau > 0$ *and* $C \equiv C(R)$ *such that*

$$\frac{1}{4}\int_0^\tau \|(\partial_t - \Delta)f(t) + f(t)^3 - f(t)\|^2_{L^2}\,\mathrm{d}t \leq C\delta.$$

Proof. We first notice that $\partial_t f(t) = \frac{1}{\tau}(y-x)$, hence $\|\partial_t f(t)\|_{L^2} \leq \frac{1}{\tau}\delta$. For the term $\Delta f(t)$ we have

$$\|\Delta f(t)\|_{L^2} \leq \|\Delta x\|_{L^2} + \|\Delta y\|_{L^2} \lesssim \|x\|_{\mathcal{B}^2_{2,2}} + \|y\|_{\mathcal{B}^2_{2,2}} \lesssim R,$$

where we use that the Besov space $\mathcal{B}^2_{2,2}$ is equivalent with the Sobolev space H^1. This is immediate from Definition 1.12 for $p = q = 2$ if we write $\|f * \eta_k\|_{L^2}$ using Plancherel's identity. For the term $f(t)^3 - f(t)$ we have

$$\begin{aligned}\|f(t)^3 - f(t)\|_{L^2} &\lesssim \|f(t)\|^3_{L^6} + \|f(t)\|_{L^2} \overset{\text{(A.5)}}{\lesssim} \|f(t)\|^3_{\mathcal{B}^0_{6,1}} + \|f(t)\|_{\mathcal{B}^0_{2,1}} \\ &\overset{\text{(A.6)},\lambda>0}{\lesssim} \|f(t)\|^3_{\mathcal{B}^{\frac{2}{3}+\lambda}_{2,2}} + \|f(t)\|_{\mathcal{B}^\lambda_{2,2}} \\ &\overset{\text{(A.1)},\lambda<\frac{1}{3}}{\lesssim} \|f(t)\|^3_{\mathcal{B}^2_{2,2}} + \|f(t)\|_{\mathcal{B}^2_{2,2}}.\end{aligned}$$

Hence for $C \equiv C(R)$

$$\frac{1}{2}\int_0^\tau \|(\partial_t - \Delta)f(t) + f(t)^3 - f(t)\|^2_{L^2}\,\mathrm{d}t \leq \frac{1}{\tau}\delta^2 + C\tau.$$

Choosing $\tau = \delta$ completes the proof. □

In the next proposition we estimate the tails of the entry time of X in a neighbourhood of ± 1 uniformly in the initial condition x. This is achieved by Proposition 5.20 and the Markov property combined with Theorem 4.1 which implies that after time $t = 1$ the process $X(\cdot; x)$ enters a compact subset of the state space with positive probability uniformly in x.

Proposition 5.22. *Let $\delta_0 > 0$ and*

$$\sigma(x) = \inf\left\{t > 0 : \min_{x_* \in \{-1,1\}} \|X(t;x) - x_*\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0\right\}.$$

For every $b > 0$ there exist $T_0 > 0$ and $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \leq \varepsilon_0$

$$\sup_{x \in \mathcal{C}^{-\alpha_0}} \mathbb{P}(\sigma(x) \geq mT_0) \leq \left(1 - \mathrm{e}^{-b/\varepsilon}\right)^m$$

for every $m \geq 1$.

Proof. By Theorem 4.1 and a simple application of Markov's inequality there exist $R_0 > 0$ such that

$$\sup_{x \in \mathcal{C}^{-\alpha_0}} \sup_{\varepsilon \in (0,1]} \mathbb{P}(\|X(1;x)\|_{\mathcal{C}^{-\alpha}} > R_0) \leq \frac{1}{2}. \tag{5.34}$$

By Proposition 5.20 for every $b > 0$ there exists $T_0 > 0$ and $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \leq \varepsilon_0$

$$\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}} \leq R_0} \mathbb{P}(\sigma(x) \geq T_0) \leq 1 - \mathrm{e}^{-b/\varepsilon}. \tag{5.35}$$

Then for every $x \in \mathcal{C}^{-\alpha_0}$ and $\varepsilon \leq \varepsilon_0$

$$\begin{aligned} \mathbb{P}(\sigma(x) \geq T_0 + 1) &\leq \mathbb{E}\left(\mathbf{1}_{\{\|X(1;x)\|_{\mathcal{C}^{-\alpha_0}} \leq R_0\}} \mathbb{P}(\sigma(X(1;x)) \geq T_0)\right) \\ &\quad + \mathbb{P}(\|X(1;x)\|_{\mathcal{C}^{-\alpha_0}} > R_0) \\ &\overset{(5.34),(5.35)}{\leq} 1 - \frac{1}{2}\mathrm{e}^{-b/\varepsilon} \end{aligned} \tag{5.36}$$

Using the Markov property successively implies for every $m \geq 1$ and $x \in \mathcal{C}^{-\alpha_0}$

$$\mathbb{P}(\sigma(x) \geq m(T_0+1)) \leq \sup_{y \in \mathcal{C}^{-\alpha_0}} \mathbb{P}(\sigma(y) \geq (T_0+1))\, \mathbb{P}(\sigma(x) \geq (m-1)(T_0+1)). \tag{5.37}$$

Combining (5.36) and (5.37) we obtain that

$$\sup_{x \in \mathcal{C}^{-\alpha_0}} \mathbb{P}(\sigma(x) \geq m(T_0 + 1)) \leq \left(1 - \frac{1}{2}\mathrm{e}^{-b/\varepsilon}\right)^m.$$

The last inequality completes the proof if we relabel b and T_0. □

Proposition 5.23. *Let $\delta_0 > 0$, $\nu_1(x)$, $\rho_1(x)$ as in Definition 5.7, $\sigma_1(x)$ as in* (5.11) *and $L(\nu_1(x), \rho_1(x); \sigma_1(x))$ as in* (5.16)*. For every $\kappa, M_0, b > 0$ there exist $T_0 > 0$ and $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \leq \varepsilon_0$*

$$\begin{aligned} &\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0} \mathbb{P}\left(\left[L(\nu_1(x), \rho_1(x); \sigma_1(x)) + (2-\kappa)\sigma_1(x) + M_0\right]^{\frac{1}{p_0}} \geq mT_0\right) \\ &\quad \leq \left(1 - \mathrm{e}^{-b/\varepsilon}\right)^m \end{aligned}$$

for every $m \geq 1$ and $p_0 \geq 1$ as in (5.16)*.*

Proof. We first condition on $\nu_1(x)$ to obtain the bound

$$\begin{aligned}
&\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}}\leq \delta_0} \mathbb{P}\left([L(\nu_1(x),\rho_1(x);\sigma_1(x)) + (2-\kappa)\sigma_1(x) + M_0]^{\frac{1}{p_0}} \geq mT_0 \right) \\
&\quad \leq \sup_{x\in\mathcal{C}^{-\alpha_0}} \underbrace{\mathbb{P}\left([L(0,\sigma(x);\sigma(x)) + (2-\kappa)\sigma(x) + M_0]^{\frac{1}{p_0}} \geq mT_0 \right)}_{=:\mathbb{P}\left(g(\sigma(x))^{\frac{1}{p_0}} \geq mT_0 \right)},
\end{aligned}$$

where $\sigma(x) = \inf\left\{ t>0 : \min_{x_*\in\{-1,1\}} \|X(t;x) - x_*\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0 \right\}$. Let $T_0 \geq 1$ to be fixed below and notice that for any $T_1 > 0$

$$\begin{aligned}
\mathbb{P}\left(g(\sigma(x))^{\frac{1}{p_0}} \geq mT_0 \right) &\leq \mathbb{P}\left(g(\sigma(x))^{\frac{1}{p_0}} \geq mT_0,\, \sigma(x) \leq mT_1 \right) \\
&\quad + \mathbb{P}(\sigma(x) \geq mT_1) \\
&\leq \mathbb{P}\left(\sum_{k=1}^{\lfloor mT_1 \rfloor} \sum_{l=0,\frac{1}{2}} L_k(l, mT_1) \geq m(T_0 - C) \right) \\
&\quad + \mathbb{P}(\sigma(x) \geq mT_1)
\end{aligned}$$

for some $C > 0$, where in the second inequality we use convexity of the mapping $g \mapsto g^{\frac{1}{p_0}}$ and the fact that $L_k(l,\sigma)$ is increasing in σ by Definition 5.9. By Proposition 5.22 we can choose $T_1 > 0$ and $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \leq \varepsilon_0$

$$\sup_{x\in\mathcal{C}^{-\alpha_0}} \mathbb{P}(\sigma(x) \geq mT_1) \leq \left(1 - \mathrm{e}^{-b/\varepsilon}\right)^m.$$

We also notice that

$$\begin{aligned}
&\mathbb{P}\left(\sum_{k=1}^{\lfloor mT_1 \rfloor} \sum_{l=0,\frac{1}{2}} L_k(l, mT_1) \geq m(T_0 - C) \right) \\
&\quad \leq \sum_{l=0,\frac{1}{2}} \mathbb{P}\left(\sum_{k=1}^{\lfloor mT_1 \rfloor} L_k(l, l+k) \geq m\left(\frac{T_0 - C}{2}\right) \right) \\
&\quad \leq \sum_{l=0,\frac{1}{2}} \exp\left\{ -cm\left(\frac{T_0 - C}{2\varepsilon}\right) \right\} \left(\mathbb{E}\mathrm{e}^{cL_1(l,1)/\varepsilon}\right)^{mT_1},
\end{aligned}$$

where in the first inequality we use that $L_k(l, mT_1) \leq L_k(l, l+k)$, for every $1 \leq k \leq \lfloor mT_1 \rfloor$, and in the second we use an exponential Chebyshev inequality, independence and equality in law of the $L_k(l, l+k)$'s. For any $T > 0$ we choose

$c \equiv c(n) > 0$ according to Proposition H.1, T_0 sufficiently large and $\varepsilon_0 \in (0,1)$ sufficiently small such that for every $\varepsilon \leq \varepsilon_0$

$$\sum_{l=0,\frac{1}{2}} \exp\left\{-cm\left(\frac{T_0 - C}{2\varepsilon}\right)\right\}\left(\mathbb{E}\mathrm{e}^{cL_1(l,1)/\varepsilon}\right)^{mT_1} \leq \mathrm{e}^{-mT/\varepsilon}.$$

Combining all the previous inequalities imply that

$$\begin{aligned}&\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0} \mathbb{P}\left([L(\nu_1(x), \rho_1(x); \sigma_1(x)) + (2-\kappa)\sigma_1(x) + M_0]^{\frac{1}{p_0}} \geq mT_0\right)\\ &\quad \leq \mathrm{e}^{-mT/\varepsilon} + \left(1 - \mathrm{e}^{-b/\varepsilon}\right)^m.\end{aligned}$$

This completes the proof if we relabel b since T is arbitrary. $\square$

5.2.3.3 Proof of Proposition 5.13

In this section we set

$$\begin{aligned}f_i(x) &:= \frac{\kappa}{2}\tau_i(x).\\ g_i(x) &:= L(\nu_i(x), \rho_i(x); \sigma_i(x)) + (2-\kappa)\sigma_i(x) + M_0.\end{aligned}$$

In this notation the random walk $S_N(x)$ is given by $\sum_{i \leq N}(f_i(x) - g_i(x))$ (see Definition 5.11).

To prove Proposition 5.13 we first consider a sequence of i.i.d. random variables $\{\tilde{f}_i\}_{i\geq 1}$ such that $\tilde{f}_1 \sim \exp(1)$. We furthermore assume that the family $\{\tilde{f}_i\}_{i\geq 1}$ is independent from both $\{f_i(x)\}_{i\geq 1}$ and $\{g_i(x)\}_{i\geq 1}$. For $\lambda > 0$ which we fix later on, we set

$$\tilde{S}_N(x) := \lambda \sum_{i \leq N} \tilde{f}_i - \sum_{i \leq N} g_i(x).$$

In the proof of Proposition 5.13 below we compare the random walk $S_N(x)$ with $\tilde{S}_N(x)$. The idea is that $\sum_{i\leq N} f_i(x)$ behaves like $\lambda \sum_{i \leq N} \tilde{f}_i$ for suitable $\lambda > 0$.

In the next proposition we estimate the new random walk $\tilde{S}_N(x)$ using stochastic dominance. In particular we assume that the family of random variables $\{g_i(x)\}_{i\geq 1}$ is stochastically dominated by a family of i.i.d. random variables $\{\tilde{g}_i\}_{i\geq 1}$ which does not depend on x and obtain a lower bound on $\mathbb{P}(-\tilde{S}_N(x) \leq u$ for every $N \geq 1)$.

From now on we denote by μ_Z the law of a random variable Z.

Proposition 5.24. *Assume that there exists a family of i.i.d. random variables $\{\tilde{g}_i\}_{i\geq 1}$, independent from both $\{g_i(x)\}_{i\geq 1}$ and $\{\tilde{f}_i\}_{i\geq 1}$, such that*

$$\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}}\leq \delta_0} \mathbb{P}(g_i(x)\geq g)\leq \mathbb{P}(\tilde{g}_i\geq g)$$

for every $g\geq 0$. Let $\tilde{S}_N=\lambda\sum_{i\leq N}\tilde{f}_i-\sum_{i\leq N}\tilde{g}_i$. Then

$$\inf_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}}\leq \delta_0} \mathbb{P}(-\tilde{S}_N(x)\leq u \textit{ for every } N\geq 1)\geq \mathbb{P}(-\tilde{S}_N\leq u \textit{ for every } N\geq 1).$$

Proof. Let

$$G_N(x,u)=\mathbb{P}(-\tilde{S}_M(x)\leq u \text{ for every } N\geq M\geq 1).$$
$$G_N(u)=\mathbb{P}(-\tilde{S}_M\leq u \text{ for every } N\geq M\geq 1).$$

We first prove that for every $N\geq 1$ and every x

$$G_N(x,u)\geq G_N(u). \tag{5.38}$$

For $N=1$ we have that

$$\begin{aligned} G_1(x,u)=\mathbb{P}(-\lambda\tilde{f}_1+g_1(x)\leq u)&=\int_0^\infty \mathbb{P}(g_1(x)\leq u+\lambda f)\,\mu_{\tilde{f}_1}(\,\mathrm{d}f)\\ &\geq \int_0^\infty \mathbb{P}(\tilde{g}_1\leq u+\lambda f)\,\mu_{\tilde{f}_1}(\,\mathrm{d}f)=\mathbb{P}(-\lambda\tilde{f}_1+\tilde{g}_1\leq u)=G_1(u). \end{aligned}$$

Let us assume that (5.38) holds for N. Let $\partial B_0=\{y:\|y-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}}=\delta_0\}$. Conditioning on $\left(\tilde{f}_1,g_1(x),X(\nu_2(x);x)\right)$ and using independence of $\tilde{f}_1$ from the joint law of $(g_1(x),X(\nu_2(x);x))$ we notice that

$$\begin{aligned} &G_{N+1}(x,u)\\ &\quad=\int_0^\infty\int_{[0,u+\lambda f]\times\partial B_0} G_N(y,u+\lambda f-g)\,\mu_{(g_1(x),X(\nu_2(x);x))}(\,\mathrm{d}g,\,\mathrm{d}y)\,\mu_{\tilde{f}_1}(\,\mathrm{d}f)\\ &\quad\overset{(5.38)}{\geq}\int_0^\infty\int_{[0,u+\lambda f]\times\partial B_0} G_N(u+\lambda f-g)\,\mu_{(g_1(x),X(\nu_2(x);x))}(\,\mathrm{d}g,\,\mathrm{d}y)\,\mu_{\tilde{f}_1}(\,\mathrm{d}f)\\ &\quad=\int_0^\infty\int_{[0,u+\lambda f]} G_N(u+\lambda f-g)\,\mu_{g_1(x)}(\,\mathrm{d}g)\,\mu_{\tilde{f}_1}(\,\mathrm{d}f). \end{aligned} \tag{5.39}$$

In the last equality above we use that $G_N(u+\lambda f-g)$ does not depend on y, hence we can drop the integral with respect to y. Let

$$H(g)=\mathbf{1}_{\{g\leq u+\lambda f\}}G_N(u+\lambda f-g).$$

Then for fixed $u, f \geq 0$, H is decreasing with respect to g. By Lemma I.1

$$\int_{[0,u+\lambda f]} G_N(u+\lambda f - g)\, \mu_{g_1(x)}(\,\mathrm{d}g) = \int H(g)\mu_{g_1(x)}(\,\mathrm{d}g) \geq \int H(g)\mu_{\tilde{g}_1}(\,\mathrm{d}g)$$
$$= \int_{[0,u+\lambda f]} G_N(u+\lambda f - g)\, \mu_{\tilde{g}_1}(\,\mathrm{d}g).$$

Integrating the last inequality with respect to f with $\mu_{\tilde{f}_1}$ and combining with (5.39) we obtain

$$G_{N+1}(x,u) \geq \int_0^\infty \int_{[0,u+\lambda f]} G_N(u+\lambda f - g)\, \mu_{\tilde{g}_1}(\,\mathrm{d}g)\, \mu_{\tilde{f}_1}(\,\mathrm{d}f) = G_{N+1}(u)$$

which proves (5.38). If we now take $N \to \infty$ in (5.38) we get for arbitrary x

$$G(x,u) \geq G(u)$$

which completes the proof. $\square$

In the next proposition we prove existence of a family of random variables $\{\tilde{g}_i\}_{i\geq 1}$ that satisfy the assumption of Proposition 5.24 and estimate their first moment.

Proposition 5.25. *There exists a family of i.i.d. random variables $\{\tilde{g}_i\}_{i\geq 1}$, independent from both $\{g_i(x)\}_{i\geq 1}$ and $\{\tilde{f}_i\}_{i\geq 1}$, such that*

$$\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0} \mathbb{P}(g_i(x) \geq g) \leq \mathbb{P}(\tilde{g}_i \geq g),$$

and furthermore for every $b > 0$ there exist $\varepsilon_0 \in (0,1)$ and $C > 0$ such that for every $\varepsilon \leq \varepsilon_0$

$$\mathbb{E}\tilde{g}_1 \leq C\mathrm{e}^{b/\varepsilon}.$$

Proof. We first notice that by the Markov property

$$\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0} \mathbb{P}(g_i(x) \geq g) \leq \sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0} \mathbb{P}(g_1(x) \geq g).$$

Let $F(g)$ be the right continuous version of the increasing function

$$1 - \sup_{x\in\mathcal{C}^{-\alpha_0}} \mathbb{P}(g_1(x) \geq g).$$

We consider a family of i.i.d. random variables such $\{\tilde{g}_i\}_{i\geq 1}$ independent from both $\{g_i(x)\}_{i\geq 1}$ and $\{\tilde{f}_i\}_{i\geq 1}$ such that $\mathbb{P}(\tilde{g}_i \leq g) = F(g)$. To estimate $\mathbb{E}\tilde{g}_1$ let $c_\varepsilon > 0$ to be fixed below. We notice that

$$\mathbb{E}\tilde{g}_1 \leq \sup_{g\geq 0} g\mathrm{e}^{-c_\varepsilon g^{\frac{1}{p_0}}} \mathbb{E}\exp\left\{c_\varepsilon \tilde{g}_1^{\frac{1}{p_0}}\right\} \leq \left(\frac{p_0 \mathrm{e}^{-1}}{c_\varepsilon}\right)^{p_0} \mathbb{E}\exp\left\{c_\varepsilon \tilde{g}_1^{\frac{1}{p_0}}\right\}. \tag{5.40}$$

For $b > 0$ we choose $T_0 > 0$ and $\varepsilon_0 \in (0,1)$ as in Proposition 5.23. Then for every $\varepsilon \leq \varepsilon_0$

$$\begin{aligned}
\mathbb{E}\exp\left\{c_\varepsilon \tilde{g}_1^{\frac{1}{p_0}}\right\} &= 1 + \int_0^\infty c_\varepsilon \mathrm{e}^{c_\varepsilon g} \mathbb{P}\left(\tilde{g}_1^{\frac{1}{p_0}} \geq g\right) \mathrm{d}g \\
&\leq 1 + \sum_{m\geq 0} \mathbb{P}\left(\tilde{g}_1^{\frac{1}{p_0}} \geq mT_0\right) \int_{mT_0}^{(m+1)T_0} c_\varepsilon \mathrm{e}^{c_\varepsilon g} \,\mathrm{d}g \\
&= 1 + \sum_{m\geq 0} \sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0} \mathbb{P}\left(g_1(x)^{\frac{1}{p_0}} \geq mT_0\right) \\
&\quad \times \int_{mT_0}^{(m+1)T_0} c_\varepsilon \mathrm{e}^{c_\varepsilon g} \,\mathrm{d}g \\
&\leq 1 + \mathrm{e}^{c_\varepsilon T_0} \sum_{m\geq 0} \mathrm{e}^{mc_\varepsilon T_0} \left(1 - \mathrm{e}^{-b/\varepsilon}\right)^m .
\end{aligned}$$

where in the last inequality we estimate $\mathbb{P}\left(g_1(x)^{\frac{1}{p_0}} \geq mT_0\right)$ using Proposition 5.23. We now choose $c_\varepsilon > 0$ such that $c_\varepsilon T_0 = \log\left(1 + \mathrm{e}^{-b/\varepsilon}\right)$. Then

$$\begin{aligned}
\mathbb{E}\exp\left\{c_\varepsilon \tilde{g}_1^{\frac{1}{p_0}}\right\} &\leq 1 + \left(1 + \mathrm{e}^{-b/\varepsilon}\right) \sum_{m\geq 0} \left(1 + \mathrm{e}^{-b/\varepsilon}\right)^m \left(1 - \mathrm{e}^{-b/\varepsilon}\right)^m \\
&\leq 1 + 2\sum_{m\geq 0} \left(1 - \mathrm{e}^{-2b/\varepsilon}\right)^m \\
&= 1 + 2\mathrm{e}^{2b/\varepsilon}.
\end{aligned}$$

Finally, by (5.40) we obtain that

$$\mathbb{E}\tilde{g}_1 \leq \left(\frac{p_0 \mathrm{e}^{-1} T_0}{\log\left(1 + \mathrm{e}^{-b/\varepsilon}\right)}\right)^{p_0} \left(1 + 2\mathrm{e}^{2b/\varepsilon}\right)$$

which completes the proof if we relabel b. $\square$

Remark 5.26. In the proof of Proposition 5.25 we use stretched exponential moments of $\tilde{g}_1$, although we only need 1st moments (see Lemma 5.28 below). This simplifies our calculations.

From now on we let $\tilde{S}_N = \lambda \sum_{i\le N} \tilde{f}_i - \sum_{i\le N} \tilde{g}_i$ for $\{\tilde{g}_i\}_{i\ge1}$ as in Proposition 5.25. In the next proposition we explicitly compute the probability

$$\mathbb{P}(-\tilde{S}_N \le 0 \text{ for every } N \ge 1).$$

The proof is essentially the same as the classical Cramér–Lundberg estimate (see [EKM97, Chapter 1.2]). We present it here for the reader's convenience.

Proposition 5.27. *For the random walk $\tilde{S}_N$ the following estimate holds,*

$$\mathbb{P}(-\tilde{S}_N \le 0 \textit{ for every } N \ge 1) = 1 - \frac{1}{\lambda}\mathbb{E}\tilde{g}_1.$$

Proof. Let $G(u) = \mathbb{P}(-\tilde{S}_N \le u \text{ for every } N \ge 1)$. Conditioning on $(\tilde{f}_1, \tilde{g}_1)$ and using independence we notice that

$$\begin{aligned}
&G(u)\\
&= \mathbb{P}\left(-\lambda\sum_{i=2}^{N}\tilde{f}_i + \sum_{i=2}^{N}\tilde{g}_i \le u + \lambda\tilde{f}_1 - \tilde{g}_1 \text{ for every } N \ge 2,\ -\lambda\tilde{f}_1 + \tilde{g}_1 \le u\right)\\
&= \int_0^\infty \int_0^{u+\lambda f} G(u+\lambda f - g)\,\mu_{\tilde{g}_1}(\,\mathrm{d}g)\,\mu_{\tilde{f}_1}(\,\mathrm{d}f)\\
&= \frac{1}{\lambda}\mathrm{e}^{u/\lambda}\int_u^\infty \mathrm{e}^{-\bar{f}/\lambda}\int_0^{\bar{f}} G(\bar{f}-g)\,\mu_{\tilde{g}_1}(\,\mathrm{d}g)\,\mathrm{d}\bar{f}
\end{aligned} \tag{5.41}$$

where in the last equality we use that $\tilde{f}_1 \sim \exp(1)$ and we also make the change of variables $\bar{f} = u + \lambda f$. This implies that $G(u)$ is differentiable with respect to u and in particular

$$\partial_{\bar{u}} G(\bar{u}) = \frac{1}{\lambda}G(\bar{u}) - \frac{1}{\lambda}\int_0^{\bar{u}} G(\bar{u}-g)\,\mu_{\tilde{g}_1}(\,\mathrm{d}g).$$

Integrating the last equation form 0 to u we obtain that

$$G(u) = G(x,0) + \frac{1}{\lambda}\int_0^u G(u-\bar{u})\,\mathrm{d}\bar{u} - \frac{1}{\lambda}\int_0^u\int_0^{\bar{u}} G(\bar{u}-g)\,\mu_{\tilde{g}_1}(\,\mathrm{d}g)\,\mathrm{d}\bar{u}. \tag{5.42}$$

Let $F(g) := \mu_{\tilde{g}_1}([0,g])$. A simple integration by parts implies

$$\begin{aligned}
&\int_0^u \int_0^{\bar{u}} G(\bar{u}-g)\,\mu_{\tilde{g}_1}(\,\mathrm{d}g)\,\mathrm{d}\bar{u} \\
&\quad = \int_0^u \left([G(\bar{u}-g)]_{g=0}^{\bar{u}} + \int_0^{\bar{u}} \partial_g G(\bar{u}-g) F(g)\,\mathrm{d}g \right) \mathrm{d}\bar{u} \\
&\quad = \int_0^u G(0)F(\bar{u})\,\mathrm{d}\bar{u} + \int_0^u \int_g^u \partial_g G(\bar{u}-g)\,\mathrm{d}\bar{u} F(g)\,\mathrm{d}g \\
&\quad = \int_0^u G(0)F(\bar{u})\,\mathrm{d}\bar{u} - \int_0^u [-G(\bar{u}-g)]_g^u\, F(g)\,\mathrm{d}g \\
&\quad = \int_0^u G(u-g)F(g)\,\mathrm{d}g. \qquad (5.43)
\end{aligned}$$

Combining (5.42) and (5.43) we get

$$G(u) = G(0) + \frac{1}{\lambda}\int_0^u G(u-\bar{u})\,\mathrm{d}\bar{u} - \frac{1}{\lambda}\int_0^u G(u-\bar{u})F(\bar{u})\,\mathrm{d}\bar{u}.$$

By taking $u \to \infty$ in the last equation and using the dominated convergence theorem and the law of large numbers we finally obtain

$$1 = G(0) + \frac{1}{\lambda}\mathbb{E}\tilde{g}_1$$

which completes the proof. □

Combining Propositions 5.24, 5.25 and 5.27 we obtain the following lemma.

Lemma 5.28. *For any $b > 0$ there exist $\varepsilon_0 \in (0,1)$ and $C > 0$ such that for every $\varepsilon \leq \varepsilon_0$*

$$\inf_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0} \mathbb{P}(-\tilde{S}_N(x) \leq 0 \textit{ for every } N \geq 1) \geq 1 - C\frac{\mathrm{e}^{b/\varepsilon}}{\lambda}.$$

Proof. By Propositions 5.24, 5.25 and 5.27 and

$$\begin{aligned}
&\inf_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0} \mathbb{P}(-\tilde{S}_N(x) \leq 0 \text{ for every } N \geq 1) \\
&\quad \geq \mathbb{P}(-\tilde{S}_N \leq 0 \text{ for every } N \geq 1) \\
&\quad = 1 - \frac{1}{\lambda}\mathbb{E}\tilde{g}_1.
\end{aligned}$$

Moreover, by Proposition 5.25 for every $b > 0$ there exist $\varepsilon_0 \in (0,1)$ and $C > 0$ such that for every $\varepsilon \leq \varepsilon_0$, $\mathbb{E}\tilde{g}_1 \leq C\mathrm{e}^{b/\varepsilon}$ which completes the proof. □

We are now ready to prove Proposition 5.13 which is the main goal of this section.

Proof of Proposition 5.13. We estimate $\mathbb{P}(S_N(x) \leq 0 \text{ for some } N \geq 1)$ in the following way,

$$\begin{aligned}
&\mathbb{P}(-S_N(x) \geq 0 \text{ for some } N \geq 1) \\
&\quad \leq \mathbb{P}\left(-\sum_{i\leq N} f_i(x) + \lambda \sum_{i\leq N} \tilde{f}_i \geq 0 \text{ for some } N \geq 1 \right) \\
&\qquad + \mathbb{P}(-\tilde{S}_N(x) \geq 0 \text{ for some } N \geq 1).
\end{aligned} \tag{5.44}$$

The second term on the right hand side can be estimated by Lemma 5.28 which provides a bound of the form

$$\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0} \mathbb{P}(-\tilde{S}_N(x) \geq 0 \text{ for some } N \geq 1) \leq C \frac{\mathrm{e}^{b/\varepsilon}}{\lambda}. \tag{5.45}$$

For the first term we notice that

$$\begin{aligned}
&\mathbb{P}\left(-\sum_{i\leq N} f_i(x) + \lambda \sum_{i\leq N} \tilde{f}_i \geq 0 \text{ for some } N \geq 1 \right) \\
&\quad \leq \sum_{N\geq 1} \mathbb{P}\left(-\sum_{i\leq N} f_i(x) + \lambda \sum_{i\leq N} \tilde{f}_i \geq 0 \right) \\
&\quad \leq \sum_{N\geq 1} \mathbb{P}\left(\exp\left\{ -\frac{1}{2\lambda} \sum_{i\leq N} f_i(x) + \frac{1}{2} \sum_{i\leq N} \tilde{f}_i \right\} \geq 1 \right).
\end{aligned}$$

By Markov's inequality, independence of $\{f_i(x)\}_{i\geq 1}$ and $\{\tilde{f}_i\}_{i\geq 1}$ and equality in law of the $\tilde{f}_i$'s the last inequality implies that

$$\begin{aligned}
&\mathbb{P}\left(-\sum_{i\leq N} f_i(x) + \lambda \sum_{i\leq N} \tilde{f}_i \geq 0 \text{ for some } N \geq 1 \right) \\
&\quad \leq \sum_{N\geq 1} \underbrace{\mathbb{E} \exp\left\{ -\frac{1}{2\lambda} \sum_{i\leq N} f_i(x) \right\}}_{=: I_N(x)} \underbrace{\left(\mathbb{E} \exp\left\{ \frac{\tilde{f}_1}{2} \right\} \right)^N}_{\leq 2^N \text{ since } \tilde{f}_1 \sim \exp(1)}.
\end{aligned} \tag{5.46}$$

Let $\varepsilon_0 \in (0,1)$ as in Proposition 5.19. Then for every $\varepsilon \le \varepsilon_0$

$$\begin{aligned}
I_N(x) &\le \left(\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0} \mathbb{E} \exp\left\{ -\frac{1}{2\lambda} f_1(x) \right\} \right)^N \\
&\le \left(\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0} \left[\mathbb{E} \exp\left\{ -\frac{1}{2\lambda} f_1(x) \right\} \mathbf{1}_{\{f_1(x) \ge \mathrm{e}^{2a_0/\varepsilon}\}} \right.\right. \\
&\qquad \left.\left. + \mathbb{P}\left(f_1(x) \le \mathrm{e}^{2a_0/\varepsilon} \right) \right] \right)^N \\
&\le \left(\mathrm{e}^{-\mathrm{e}^{2a_0/\varepsilon}/2\lambda} + \mathrm{e}^{-3a_0/\varepsilon} \right)^N ,
\end{aligned}$$

where in the first inequality we use the Markov property and in the last we use Proposition 5.19. If we choose $\frac{1}{2\lambda} = \mathrm{e}^{-(2a_0-b)/\varepsilon}$ and choose $\varepsilon_0 \in (0,1)$ even smaller the last inequality implies that for every $\varepsilon \le \varepsilon_0$

$$\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0} I_N(x) \le \left(\mathrm{e}^{-\mathrm{e}^{b/\varepsilon}} + \mathrm{e}^{-3a_0/\varepsilon} \right)^N \le \mathrm{e}^{-5a_0 N/2\varepsilon}.$$

Combining with (5.46) we find $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \le \varepsilon_0$

$$\begin{aligned}
&\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0} \mathbb{P}\left(-\sum_{i \le N} f_i(x) + \lambda \sum_{i \le N} \tilde{f}_i \ge 0 \text{ for some } N \ge 1 \right) \\
&\quad \le \sum_{N \ge 1} \mathrm{e}^{-5a_0 N/2\varepsilon} 2^N \le \sum_{N \ge 1} \mathrm{e}^{-2a_0 N/\varepsilon} = \frac{\mathrm{e}^{-2a_0/\varepsilon}}{1-\mathrm{e}^{-2a_0/\varepsilon}}. \qquad (5.47)
\end{aligned}$$

Finally (5.44), (5.45) and (5.47) imply that

$$\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0} \mathbb{P}(-S_N(x) \ge 0 \text{ for some } N \ge 1) \le C \frac{\mathrm{e}^{b/\varepsilon}}{\mathrm{e}^{(2a_0-b)/\varepsilon}} + \frac{\mathrm{e}^{-2a_0/\varepsilon}}{1-\mathrm{e}^{-2a_0/\varepsilon}}$$

which completes the proof since b is arbitrary. □

5.3 Applications to Eyring–Kramers Law

In this section we consider the spatial Galerkin approximation $X^N(\cdot;x)$ of $X(\cdot;x)$ given by

$$\begin{aligned}
(\partial_t - \Delta) X^N &= -\Pi_N \left(\left(X^N\right)^3 - X^N - 3\varepsilon \Re_N X^N \right) + \sqrt{2\varepsilon}\xi_N \\
X^N\big|_{t=0} &= x_N
\end{aligned} \qquad (5.48)$$

where Π_N is the projection on $\{f \in L^2 : f(z) = \sum_{|k|\le N} \hat f(k) L^{-2} \mathrm{e}^{2\pi \mathrm{i} k\cdot z/L}\}$, $\xi_N = \Pi_N \xi$, $x_N = \Pi_N x$ and $\Re_N$ is as in (2.9). Here for $k \in \mathbb{Z}^2$ we set $|k| = |k_1| \vee |k_2|$. In this notation we have that $\Pi_N f = f * D_N$, where D_N is the 2-dimensional square Dirichlet kernel given by $D_N(z) = \sum_{|k|\le N} L^{-2}\mathrm{e}^{2\pi \mathrm{i} k\cdot z/L}$.

To treat (5.48) we write $X^N(\cdot;x) = v^N(\cdot;x) + \varepsilon^{\frac12} \uparrow^N(\cdot;x)$ for

$$\begin{aligned}&\big(\partial_t - (\Delta - 1)\big)\uparrow^N = \sqrt{2}\xi_N\\ &\uparrow^N(0) = 0.\end{aligned}$$

Then $v^N(\cdot;x)$ solves

$$\begin{aligned}(\partial_t - \Delta)\, v^N &= -\Pi_N (v^N)^3 + v^N - \Pi_N\left(3(v^N)^2 \varepsilon^{\frac12}\uparrow^N + 3v^N \varepsilon \mathbf{V}^N + \varepsilon^{\frac32}\boldsymbol{\Psi}^N\right)\\ &\quad + 2\varepsilon^{\frac12}\uparrow^N \qquad (5.49)\\ v^N\big|_{t=0} &= x_N\end{aligned}$$

where $\mathbf{V}^N = \left(\uparrow^N\right)^2 - \Re_N$ and $\boldsymbol{\Psi}^N = \left(\uparrow^N\right)^3 - 3\Re_N \uparrow^N$.

For $\delta \in (0,1/2)$ and $\alpha > 0$ we define the symmetric subsets A and B of $\mathcal{C}^{-\alpha}$ by

$$A := \left\{f \in \mathcal{C}^{-\alpha} : \bar f \in [-1-\delta, -1+\delta],\ f - \bar f \in D_\perp\right\} \qquad (5.50)$$

$$B := \left\{f \in \mathcal{C}^{-\alpha} : \bar f \in [1-\delta, 1+\delta],\ f - \bar f \in D_\perp\right\} \qquad (5.51)$$

where $D_\perp$ is a closed ball of radius δ in $\mathcal{C}^{-\alpha}$ and $\bar f = L^{-2}\langle f, 1\rangle$. If necessary we write $A(\alpha;\delta)$ and $B(\alpha;\delta)$ to denote the specific value of the parameters α and δ. Last for $x \in A$ we define

$$\tau_B(X^N(\cdot;x)) := \inf\left\{t > 0 : X^N(t;x) \in B\right\}$$

and

$$\tau_B(X(\cdot;x)) := \inf\left\{t > 0 : X(t;x) \in B\right\}.$$

For $k \in \mathbb{Z}^2$ let

$$\lambda_k := \left(\frac{2\pi|k|}{L}\right)^2 - 1 \text{ and } \quad \nu_k := \left(\frac{2\pi|k|}{L}\right)^2 + 2 = \lambda_k + 3.$$

The sequences $\{\lambda_k\}_{k\in\mathbb{Z}^2}$ and $\{\nu_k\}_{k\in\mathbb{Z}^2}$ are the eigenvalues of the operators $-\Delta - 1$ and $-\Delta + 2$ endowed with periodic boundary conditions.

The next theorem is essentially [BDGW17, Theorem 2.3].

Theorem 5.29 ([BDGW17, Theorem 2.3]). *Let $0 < L < 2\pi$. For every $\alpha > 0$, $\delta \in (0,1/2)$ and $\varepsilon \in (0,1)$ there exists a sequence $\{\mu_{\varepsilon,N}\}_{N\geq 1}$ of probability measures concentrated on ∂A such that*

$$\begin{aligned}
&\limsup_{N\to\infty} \int \mathbb{E}\tau_B(X^N(\cdot;x))\,\mu_{\varepsilon,N}(\mathrm{d}x) \\
&\quad\leq \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2}\frac{|\lambda_k|}{\nu_k}\exp\left\{\frac{\nu_k-\lambda_k}{\lambda_k+2}\right\}}\mathrm{e}^{(V(0)-V(-1))/\varepsilon}\left(1+c_+\sqrt{\varepsilon}\right) \\
&\liminf_{N\to\infty} \int \mathbb{E}\tau_B(X^N(\cdot;x))\,\mu_{\varepsilon,N}(\mathrm{d}x) \\
&\quad\geq \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2}\frac{|\lambda_k|}{\nu_k}\exp\left\{\frac{\nu_k-\lambda_k}{\lambda_k+2}\right\}}\mathrm{e}^{(V(0)-V(-1))/\varepsilon}\left(1-c_-\varepsilon\right)
\end{aligned} \tag{5.52}$$

where the constants c_+ and c_- are uniform in ε.

Proof. The proof of (5.52) is given in [BDGW17, Sections 4 and 5], but the following should be modified.

- In [BDGW17], the sets A and B are defined as in (5.50) and (5.51) with $D_\perp$ replaced by a ball in H^s for $s < 0$. The explicit form of $D_\perp$ is only used in [BDGW17, Lemma 5.9]. There the authors consider the 0-mean Gaussian measure $\gamma_0^\perp$ with quadratic form $\frac{1}{2\varepsilon}\left(\|\nabla f\|_{L^2}^2 - \|f-\bar{f}\|_{L^2}^2\right)$, and prove that $D_\perp$ has probability bounded from below by $1-c\varepsilon^2$. Here we assume that $D_\perp$ is a ball in $\mathcal{C}^{-\alpha}$. To obtain the same estimate for this set, we first notice that the random field f associated with the measure $\gamma_0^\perp$ satisfies

$$\mathbb{E}\langle f, L^{-2}\mathrm{e}^{2\mathrm{i}\pi k\cdot/L}\rangle \lesssim \frac{\varepsilon\log\varepsilon^{-1}\log\lambda_k}{1+\lambda_k},$$

  for every $k\in\mathbb{Z}^2$, where the explicit constant depends on L. This decay of the Fourier modes of f and [MWX17, Proposition 3.6] imply that the measure $\gamma_0^\perp$ is concentrated in $\mathcal{C}^{-\alpha}$, for every $\alpha > 0$, which in turn implies [BDGW17, Lemma 5.9] for the set $D_\perp$ considered here.

- In [BDGW17], the authors consider (5.48) with $\Re_N$ replaced by

$$C_N = \frac{1}{L^2}\sum_{|k|\leq N}\frac{1}{|\lambda_k|}$$

and obtain (5.52) with the pre-factor given by

$$\frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2}\frac{|\lambda_k|}{\nu_k}\exp\left\{\frac{\nu_k-\lambda_k}{\lambda_k}\right\}} = \lim_{N\to\infty}\frac{2\pi}{|\lambda_0|}\sqrt{\prod_{|k|\le N}\frac{|\lambda_k|}{\nu_k}}\exp\left\{\frac{3L^2C_N}{2}\right\}.$$

In our case one can check by (2.9) that $\Re_N$ is given by

$$\Re_N = \frac{1}{L^2}\sum_{|k|\le N}\frac{1}{|\lambda_k+2|}.$$

According to [BDGW17, Remark 2.5] this choice of renormalisation constant modifies [BDGW17, Theorem 2.3] by multiplying the pre-factor there with

$$\exp\left\{-3L^2\lim_{N\to\infty}(\Re_N - C_N)/2\lambda_0\right\}.$$

□

Remark 5.30. The finite dimensional measure $\mu_{\varepsilon,N}$ in (5.52) is given by

$$\mu_{\varepsilon,N}(\,\mathrm{d}f) = \frac{1}{\mathrm{cap}_A(B)}\mathrm{e}^{-V(\Pi_N f)/\varepsilon}\rho_{A,B}(\,\mathrm{d}f),$$

where $\rho_{A,B}$ is a probability measure concentrated on ∂A, called the equilibrium measure, and $\mathrm{cap}_A(B)$ is a normalisation constant. Under this measure and the assumption that the sets A and B are symmetric, the integrals appearing in (5.52) can be rewritten using potential theory as

$$\int \mathbb{E}\tau_B(X^N(\cdot;x))\,\mu_{\varepsilon,N}(\,\mathrm{d}x) = \frac{1}{2\mathrm{cap}_A(B)}\int_{\mathbb{R}^{(2N+1)^2}}\mathrm{e}^{-V(\Pi_N f)/\varepsilon}\,\mathrm{d}f.$$

This formula is derived in [BDGW17, Section 3] and it is then analysed to obtain (5.52).

In the next theorem, which is the main result of this section, we generalise (5.52) for the limiting process $X(\cdot;x)$ for fixed initial condition x in a suitable neighbourhood of -1. By symmetry the same results holds if we swap the neighbourhoods of -1 and 1 below.

Theorem 5.31. *There exist $\delta_0 > 0$ such that the following holds. For every $\alpha\in(0,\alpha_0)$ and $\delta\in(0,\delta_0)$ there exist $c_+, c_- > 0$ and $\varepsilon_0\in(0,1)$ such that for every*

$\varepsilon \le \varepsilon_0$

$$\begin{aligned}
&\sup_{x\in A(\alpha_0;\delta)} \mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;x)) \\
&\quad\le \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2}\frac{|\lambda_k|}{\nu_k}\exp\left\{\frac{\nu_k-\lambda_k}{\lambda_k+2}\right\}}\mathrm{e}^{(V(0)-V(-1))/\varepsilon}\left(1+c_+\sqrt{\varepsilon}\right). \\
&\inf_{x\in A(\alpha_0;\delta)} \mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;x)) \\
&\quad\ge \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2}\frac{|\lambda_k|}{\nu_k}\exp\left\{\frac{\nu_k-\lambda_k}{\lambda_k+2}\right\}}\mathrm{e}^{(V(0)-V(-1))/\varepsilon}\left(1-c_-\varepsilon\right).
\end{aligned} \tag{5.53}$$

Proof. See Section 5.3.3. □

To prove this theorem we first fix $\alpha \in (0,\alpha_0)$ and pass to the limit as $N \to \infty$ in (5.52) to prove a version of (5.53) where the initial condition x is averaged with respect to a measure μ_ε concentrated on a closed ball with respect to the weaker topology $\mathcal{C}^{-\alpha_0}$ (see Proposition 5.36). This measure is the weak limit, up to a subsequence, of the measures $\mu_{\varepsilon,N}$ in Theorem 5.29. We then use our "exponential loss of memory" result, Theorem 5.3, to pass from averages of initial conditions with respect to the limiting measure μ_ε to fixed initial conditions.

The rest of this section is structured as follows. In Section 5.3.1 we prove convergence of the Galerkin approximations $X^N(\cdot;x_N)$ and obtain estimates uniform in the initial condition x and the regularisation parameter N. In Section 5.3.2 we prove uniform integrability of the stopping times $\tau_B(X(\cdot;x))$ and pass to the limit as $N \to \infty$ in (5.52). Finally in Section 5.3.3 we prove Theorem 5.31.

5.3.1 Convergence and A Priori Estimates of the Galerkin Scheme

In the next proposition we use convergence of the stochastic objects $\curlyvee^N$ (see 2.3) to prove convergence of $X^N(\cdot;x_N)$ to $X(\cdot;x)$ in $C([0,T];\mathcal{C}^{-\alpha})$. This is a technical result and the proof is given in the Appendix.

Proposition 5.32. *Let $\aleph \subset \mathcal{C}^{-\alpha_0}$ be bounded and assume that for every $x \in \aleph$, there exists a sequence $\{x_N\}_{N\ge 1}$ such that $x_N \to x$ uniformly in x. Then for every*

$\alpha \in (0, \alpha_0)$ *and* $0 < s < T$

$$\lim_{N\to\infty} \sup_{x\in\aleph} \sup_{t\in[s,T]} \|X^N(t;x_N) - X(t;x)\|_{\mathcal{C}^{-\alpha}} = 0$$

in probability.

Proof. See Appendix J. $\square$

The next proposition provides a bound for $X^N(\cdot;x)$ uniformly in the initial condition x and the regularisation parameter N in the $\mathcal{B}^{-\alpha}_{2,2}$ norm, for $0 < \alpha < \alpha_0$. This result has already been established in Theorem 4.1 for the limiting process $X(\cdot;x)$ in the $\mathcal{C}^{-\alpha}$ norm. There (5.6) is tested with $v(\cdot;x)^{p-1}$, for $p \geq 2$ even, to bound $\|v(\cdot;x)\|_{L^p}$ by using the “good” sign of the non-linear term $-v^3$. In the case of (5.49) this argument allows us to bound $\|v^N(\cdot;x)\|_{L^p}$ for $p = 2$ only, because of the projection Π_N in front of the non-linearity.

Proposition 5.33. *For every* $\alpha \in (0, \alpha_0]$ *and* $p \geq 1$ *we have that*

$$\sup_{N\geq 1} \sup_{x\in\mathcal{C}^{-\alpha_0}} \sup_{t\leq 1} t^{\frac{p}{2}} \mathbb{E}\|X^N(s;x)\|^p_{\mathcal{B}^{-\alpha}_{2,2}} < \infty. \tag{5.54}$$

Proof. Proceeding exactly as in the proof of Proposition 3.10 we first show that there exist $\alpha \in (0,1)$ and $p_n \geq 1$ such that for every $t \in (0,1)$

$$\|v^N(t;x)\|^2_{L^2} \lesssim t^{-1} \vee \left(\sum_{n=1}^{3} t^{-\alpha'(n-1)p_n} \sup_{s\leq t} s^{\alpha'(n-1)p_n} \|\varepsilon^{\frac{n}{2}} \Psi^{:n:,N}(s)\|^{p_n}_{\mathcal{C}^{-\alpha}} \right)^{\frac{1}{2}} \tag{5.55}$$

for every $\alpha' \in (0,1)$, uniformly in $x \in \mathcal{C}^{-\alpha_0}$. We then proceed as in the proof of Theorem 4.1 and use (5.55) to prove (5.54). The only difference is that here we use the norm $\|\cdot\|_{\mathcal{B}^{-\alpha}_{2,2}}$ and the embedding $L^2 \hookrightarrow \mathcal{B}^{-\alpha}_{2,2}$ on the level of $v^N(\cdot;x)$ together with the fact that

$$\sup_{N\geq 1} \mathbb{E} \left(\sup_{t\leq 1} t^{(n-1)\alpha'} \|\Psi^{:n:,N}(t)\|_{\mathcal{C}^{-\alpha}} \right)^p < \infty$$

for every $\alpha, \alpha' > 0$ and $p \geq 1$, which is immediate from Propositions 2.2 and 2.3. $\square$

5.3.2 Passing to the Limit

In this section we pass to the limit as $N \to \infty$ in (5.52) using uniform integrability of the stopping time $\tau_B(X^N(\cdot;x))$. To obtain uniform integrability we prove exponential moment bounds for $\tau_B(X^N(\cdot;x))$ uniformly in the initial condition $x \in \mathcal{C}^{-\alpha_0}$ and the regularisation parameter N. We first bound $\mathbb{P}\left(\tau_B(X^N(\cdot;x)) \geq 1\right)$ using a support theorem and a strong dissipative bound for $X^N(\cdot;x)$ in $\mathcal{C}^{-\alpha}$. A support theorem for the limiting process $X(\cdot;x)$ has already been established in Corollary 4.18. To use it for $X^N(\cdot;x)$, we combine it with the convergence result in Proposition 5.32. To obtain a strong dissipative bound for $X^N(\cdot;x)$ in $\mathcal{C}^{-\alpha}$ we first use Proposition 5.33 which implies the bound in $\mathcal{B}^{-\alpha}_{2,2}$ and then use Proposition K.2 to pass from the $\mathcal{B}^{-\alpha}_{2,2}$ norm to the $\mathcal{C}^{-\alpha}$ norm.

Proposition 5.34. *For every $\alpha \in (0,\alpha_0)$, $\delta \in (0,1/2)$ and $\varepsilon \in (0,1)$ there exist $c_0 \in (0,1)$ and $N_0 \geq 1$ such that for every $N \geq N_0$*

$$\sup_{x \in \mathcal{C}^{-\alpha_0}} \mathbb{P}\left(\tau_B(X^N(\cdot;x)) \geq 1\right) \leq c_0.$$

Proof. Let $\alpha \in (0,\alpha_0)$ and let $\aleph$ be a compact subset of $\mathcal{C}^{-\alpha_0}$ which we fix below. Using the Markov property

$$\mathbb{P}(\tau_B(X^N(\cdot;x)) \geq 1) \leq \sup_{y\in\aleph} \mathbb{P}(\tau_B(X^N(\cdot;y)) \geq 1/2)\,\mathbb{P}(X^N(1/2;x) \in \aleph) + \mathbb{P}(X^N(1/2;x) \notin \aleph).$$

The proof is complete if for every $N \geq N_0$

$$\sup_{y\in\aleph} \mathbb{P}(\tau_B(X^N(\cdot;y)) \geq 1/2) < 1, \quad \sup_{x\in\mathcal{C}^{-\alpha_0}} \mathbb{P}(X^N(1/2;x) \notin \aleph) < 1. \tag{5.56}$$

We notice that there exists $\delta' > 0$ such that for any $y \in \aleph$

$$\begin{aligned}\mathbb{P}(\tau_B(X^N(\cdot;y)) \leq 1/2) \geq \mathbb{P}(X^N(1/2;y) \in B) &\geq \mathbb{P}\left(X(1/2;y) \in B_{\mathcal{C}^{-\alpha}}(1;\delta')\right)\\ &\quad - \mathbb{P}\left(\|X^N(1/2;y) - X(1/2;y)\|_{\mathcal{C}^{-\alpha}} \geq \delta'\right). \end{aligned}\tag{5.57}$$

Here we use that if $\|X(1/2;y) - 1\|_{\mathcal{C}^{-\alpha}}, \|X^N(1/2;y) - X(1/2;y)\|_{\mathcal{C}^{-\alpha}} \leq \delta'$, then $X^N(1/2;y) \in B$ for δ' sufficiently small. By the support theorem Corollary 4.18 there exists $c_1 \equiv c_1(\delta,\varepsilon) > 0$ such that

$$\inf_{y\in\aleph} \mathbb{P}\left(X(1/2;y) \in B_{\mathcal{C}^{-\alpha}}(1;\delta')\right) \geq c_1. \tag{5.58}$$

On the other hand Proposition 5.32 implies that

$$X^N(1/2;y) \to X(1/2;y)$$

in $\mathcal{C}^{-\alpha}$ in probability uniformly in $y \in \aleph$. Hence there exists $N_0 \geq 1$ such that for every $N \geq N_0$

$$\sup_{y\in\aleph} \mathbb{P}\left(\|X^N(1/2;y) - X(1/2;y)\|_{\mathcal{C}^{-\alpha}} \geq \delta/2\right) \leq c_1/2. \tag{5.59}$$

Plugging (5.58) and (5.59) in (5.57) implies the first bound in (5.56).

We now prove the second bound in (5.56). By the Markov inequality for every $R > 0$

$$\mathbb{P}\left(\|X^N(1/4;x)\|_{\mathcal{B}^{-\alpha}_{2,2}} \geq R\right) \leq \frac{1}{R}\mathbb{E}\|X^N(1/4;x)\|_{\mathcal{B}^{-\alpha}_{2,2}}.$$

By (5.54) the expectation on the right hand side of the last inequality is uniformly bounded over $x \in \mathcal{C}^{-\alpha_0}$ and $N \geq 1$. Thus choosing $R > 0$ large enough

$$\sup_{x\in\mathcal{C}^{-\alpha_0}} \mathbb{P}\left(\|X^N(1/4;x)\|_{\mathcal{B}^{-\alpha}_{2,2}} \geq R\right) \leq \frac{1}{2}. \tag{5.60}$$

By Proposition K.2 for every $K, R > 0$ there exist $C \equiv C(K,R)$ such that

$$\sup_{\|y\|_{\mathcal{B}^{-\alpha}_{2,2}}\leq R} \mathbb{P}\left(\|X^N(1/4;y)\|_{\mathcal{C}^{-\alpha}} \geq C\right) \leq \mathbb{P}\left(\sup_{t\leq 1} t^{(n-1)\alpha'}\|\varepsilon^{\frac{n}{2}} \Psi^{n,N}(t)\|_{\mathcal{C}^{-\alpha}} \geq K\right).$$

Choosing K sufficiently large, combining the last inequality with Propositions 2.2 and 2.3 and using the Markov inequality imply that

$$\sup_{\|y\|_{\mathcal{B}^{-\alpha}_{2,2}}\leq R} \mathbb{P}\left(\|X^N(1/4;y)\|_{\mathcal{C}^{-\alpha}} \geq C\right) \leq \frac{1}{2}. \tag{5.61}$$

Using the Markov property and (5.60) and (5.61) we get for arbitrary $x \in \mathcal{C}^{-\alpha_0}$

$$\begin{aligned}
&\mathbb{P}\left(\|X^N(1/2;x)\|_{\mathcal{C}^{-\alpha}} \geq C\right) \\
&\quad\leq \mathbb{P}\left(\|X^N(1/4;x)\|_{\mathcal{B}^{-\alpha}_{2,2}} \leq R\right) \sup_{y\in\mathcal{B}^{-\alpha}_{2,2}} \mathbb{P}(\|X^N(1/4;y)\|_{\mathcal{C}^{-\alpha}} \geq C) \\
&\quad+ \mathbb{P}\left(\|X^N(1/4;x)\|_{\mathcal{B}^{-\alpha}_{2,2}} \geq R\right) \\
&\quad\leq \frac{3}{4}.
\end{aligned}$$

We finally notice that for every $\alpha < \alpha_0$ the set $\aleph = \{f \in \mathcal{C}^{-\alpha_0} : \|f\|_{\mathcal{C}^{-\alpha}} \leq C\}$ is compact in $\mathcal{C}^{-\alpha_0}$ which implies the second bound in (5.56). $\square$

In the next corollary we use Proposition 5.34 to prove exponential moments for the stopping time $\tau_B(X^N(\cdot;x))$.

Corollary 5.35. *For every $\delta > 0$ and $\varepsilon \in (0,1)$ there exist $\eta_0 > 0$ and $N_0 \geq 1$ such that*

$$\sup_{N \geq N_0} \sup_{x \in \mathcal{C}^{-\alpha_0}} \mathbb{E} \exp\{\eta_0 \tau_B(X^N(\cdot;x))\} < \infty.$$

Proof. By the Markov property we have that

$$\mathbb{P}(\tau_B(X^N(\cdot;x)) \geq k+1) \leq \sup_{y \in \mathcal{C}^{-\alpha_0}} \mathbb{P}(\tau_B(X^N(\cdot;y)) \geq 1)\, \mathbb{P}(\tau_B(X^N(\cdot;x)) \geq k).$$

Iterating this inequality and using Proposition 5.34 we obtain that

$$\sup_{x \in \mathcal{C}^{-\alpha_0}} \mathbb{P}(\tau_B(X^N(\cdot;x)) \geq k+1) \leq c_0^{k+1}.$$

Then

$$\begin{aligned}
\mathbb{E} \exp\{\eta_0 \tau_B(X^N(\cdot;x))\} &= 1 + \int_0^\infty \eta_0 \mathrm{e}^{\eta_0 t} \mathbb{P}(\tau_B(X^N(\cdot;x)) \geq t)\, \mathrm{d}t \\
&\leq 1 + \sum_{k=0}^\infty \mathbb{P}(\tau_B(X^N(\cdot;x)) \geq k) \int_k^{k+1} \eta_0 \mathrm{e}^{\eta_0 t}\, \mathrm{d}t \\
&\leq 1 + \mathrm{e}^{\eta_0} \sum_{k=0}^\infty \mathrm{e}^{\eta_0 k} c_0^k
\end{aligned}$$

and the proof is complete if we choose $\eta_0 < \log c_0^{-1}$. □

In the next proposition we pass to the limit as $N \to \infty$ in (5.52). Here we use Corollary 5.35, which implies uniform integrability of $\tau_B(X^N(\cdot;x))$, and the weak convergence of the measures $\mu_{\varepsilon,N}$.

Proposition 5.36. *For every $\alpha \in (0, \alpha_0)$, $\delta \in (0, 1/2)$ except possibly a countable subset, and $\varepsilon \in (0,1)$ there exists a probability measure $\mu_\varepsilon \in \mathcal{M}_1(A(\alpha_0;\delta))$ such that*

$$\begin{aligned}
&\int \mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;x))\, \mu_\varepsilon(\mathrm{d}x) \\
&\quad \leq \frac{2\pi}{|\lambda_0|} \sqrt{\prod_{k \in \mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k} \exp\left\{\frac{\nu_k - \lambda_k}{\lambda_k + 2}\right\}} \mathrm{e}^{(V(0)-V(-1))/\varepsilon} \left(1 + c_+\sqrt{\varepsilon}\right) \\
&\int \mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;x))\, \mu_\varepsilon(\mathrm{d}x) \\
&\quad \geq \frac{2\pi}{|\lambda_0|} \sqrt{\prod_{k \in \mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k} \exp\left\{\frac{\nu_k - \lambda_k}{\lambda_k + 2}\right\}} \mathrm{e}^{(V(0)-V(-1))/\varepsilon} \left(1 - c_-\varepsilon\right)
\end{aligned} \tag{5.62}$$

where the constants c_+ and c_- are uniform in ε.

Proof. We only prove the upper bound in 5.62. The lower bound follows similarly.

Let $\alpha \in (0, \alpha_0)$ and $\delta \in (0, 1/2)$. Using the compact embedding $\mathcal{C}^{-\alpha} \hookrightarrow \mathcal{C}^{-\alpha_0}$ (see Proposition A.4), for any $\alpha < \alpha_0$, we have that $A(\alpha;\delta) \subset A(\alpha_0;\delta)$. Let $\{\mu_{\varepsilon,N}\}_{N\geq 1}$ be the family of probability measures in (5.52). Using again the compact embedding $\mathcal{C}^{-\alpha} \hookrightarrow \mathcal{C}^{-\alpha_0}$, for any $\alpha < \alpha_0$, this family is trivially tight since it is concentrated on $\partial A(\alpha;\delta)$. Hence there exists $\mu_\varepsilon \in \mathcal{M}_1\left(A(\alpha_0;\delta)\right)$ such that $\mu_{\varepsilon,N} \overset{\text{weak}}{\to} \mu_\varepsilon$ up to a subsequence.

By Skorokhod's represantation theorem (see [DPZ92, Theorem 2.4]) there exist a probability space $(\Omega_\mu, \mathcal{F}_\mu, \mathbb{P}_\mu)$ and random variables $\{x_N\}_{N\geq 1}$ and x taking values in $A(\alpha_0;\delta)$ such that $x_N \overset{\text{law}}{=} \mu_N$, $x \overset{\text{law}}{=} \mu_\varepsilon$ and $x_N \to x$ $\mathbb{P}_{\mu_\varepsilon}$-almost surely in $\mathcal{C}^{-\alpha_0}$. If we denote by $\mathbb{E}_{\mathbb{P}\otimes\mathbb{P}_{\mu_\varepsilon}}$ the expectation of the probability measure $\mathbb{P}\otimes\mathbb{P}_{\mu_\varepsilon}$, we have that

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}\otimes\mathbb{P}_{\mu_\varepsilon}} \tau_{B(\alpha;\delta)}(X^N(\cdot;x_N)) &= \int \mathbb{E}\tau_{B(\alpha;\delta)}(X^N(\cdot;x))\,\mu_N(\,\mathrm{d}x) \\
\mathbb{E}_{\mathbb{P}\otimes\mathbb{P}_{\mu_\varepsilon}} \tau_{B(\alpha;\delta)}(X(\cdot;x)) &= \int \mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;x))\,\mu_\varepsilon(\,\mathrm{d}x).
\end{aligned} \tag{5.63}$$

By Proposition 5.32 $X^N(\cdot;x_N)$ converges to $X(\cdot;x)$ $\mathbb{P}\otimes\mathbb{P}_{\mu_\varepsilon}$-almost surely on compact time intervals of $(0,\infty)$ up to a subsequence. Let

$$\mathtt{L} = \left\{\delta \in (0,1/2) : \mathbb{P}\left(\tau_{B(\alpha;\delta)}(\cdot) \text{ is discontinuous on } X(\cdot;x)\right) > 0\right\}$$

and notice that if M denotes the mapping

$$t \mapsto |L^{-2}\langle X(t;x),1\rangle - 1| \vee \|X(t;x) - L^{-2}\langle X(t;x),1\rangle\|_{\mathcal{C}^{-\alpha}}$$

then

$$\mathtt{L} \subset \left\{\delta \in (0,1/2) : \mathbb{P}\left(M \text{ has a local minimum at height } \delta\right) > 0\right\}.$$

As in [MW17c, Proof of Theorem 6.1] the last set is at most countable, hence $\tau_{B(\alpha;\delta)}(X^N(\cdot;x_N)) \to \tau_{B(\alpha;\delta)}(X(\cdot;x))$, $\mathbb{P}\otimes\mathbb{P}_{\mu_\varepsilon}$-almost surely up to a subsequence, except possibly a countable number of $\delta \in (0,1/2)$.

By Corollary 5.35 the family $\{\tau_{B(\alpha;\delta)}(X^N(\cdot;x))\}_{N\geq N_0}$ is uniformly integrable. Hence by Vitali's convergence theorem (see [Bog07, Theorem 4.5.4]) we obtain that

$$\mathbb{E}_{\mathbb{P}\otimes\mathbb{P}_{\mu_\varepsilon}} \tau_{B(\alpha;\delta)}(X^N(\cdot;x_N)) \to \mathbb{E}_{\mathbb{P}\otimes\mathbb{P}_{\mu_\varepsilon}} \tau_{B(\alpha;\delta)}(X(\cdot;x)).$$

Combining with (5.52) and (5.63) the proof of the upper bound is complete. □

5.3.3 An Eyring-Kramers Law

In this section we combine Proposition 5.36 and Theorem 5.3 to prove Theorem 5.31. The idea we use here was first implemented in the 1-dimensional case in [BG13]. Generally speaking, if we restrict ourselves on the event where the first transition happens after the "exponential loss of memory", $\tau_{B(\alpha;\delta)}(X(\cdot;x))$ behaves like $\int \tau_{B(\alpha;\delta)}(X(\cdot;x))\,\mu_\varepsilon(\,\mathrm{d}x)$ for $x \in A(\alpha_0;\delta)$. The probability of this event is quantified by Theorem 5.3 and Proposition 5.37. On the complement of this event the transition time $\tau_{B(\alpha;\delta)}(X(\cdot;x))$ is estimated using Proposition 5.38.

In the next proposition we prove that the first transition from a neighbourhood of -1 to a neighbourhood of 1 happens only after some time $T_0 > 0$ with overwhelming probability. This is a large deviation event which can be estimated using continuity of X with respect to the initial condition x and the stochastic objects $\left\{\varepsilon^{\frac{n}{2}} \bigvee^{n}\right\}_{n\le 3}$. We sketch the proof for completeness.

Proposition 5.37. *For every* $\alpha \in (0,\alpha_0)$ *and* $\delta \in (0,1/2)$ *there exist* $a_0, \delta_0, T_0 > 0$ *and* $\varepsilon_0 \in (0,1)$ *such that for every* $\varepsilon \le \varepsilon_0$

$$\sup_{\|x-(-1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0} \mathbb{P}(\tau_{B(\alpha;\delta)}(X(\cdot;x)) \le T_0) \le \mathrm{e}^{-a_0/\varepsilon}.$$

Proof. We first notice that for $\|x-(-1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0$

$$\mathbb{P}(\tau_{B(\alpha;\delta)}(X(\cdot;x)) \ge T_0) \ge \mathbb{P}\left(\sup_{t\le T_0} \|X(t;x)-(-1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_1\right)$$

for some $\delta_1 > 0$. Using continuity of X with respect to x and the stochastic objects $\left\{\varepsilon^{\frac{n}{2}} \bigvee^{n}\right\}_{n\le 3}$, the last probability can be estimated from below uniformly in $\|x-(-1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0$, for δ_0 sufficiently small, by

$$\mathbb{P}\left(\sup_{t\le T_0} \|X(t;x)-(-1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_1\right) \ge \mathbb{P}\left(\sup_{t\le T_0} (t\wedge 1)^{-(n-1)\alpha'} \|\varepsilon^{\frac{n}{2}} \bigvee^{n}(t)\|_{\mathcal{C}^{-\alpha}} \le \delta_2\right)$$

for some $\delta_2 > 0$. Last by Proposition H.1 we find $a_0 > 0$ and $\varepsilon_0 \in (0,1)$ such that for ever $\varepsilon \le \varepsilon_0$

$$\mathbb{P}\left(\sup_{t\le T_0} (t\wedge 1)^{-(n-1)\alpha'} \|\varepsilon^{\frac{n}{2}} \bigvee^{n}(t)\|_{\mathcal{C}^{-\alpha}} \le \delta_2\right) \ge 1-\mathrm{e}^{-a_0/\varepsilon}$$

which completes the proof. $\square$

In the next proposition we estimate the second moment of the transition time $\tau_{B(\alpha;\delta)}(X(\cdot;x))$ using the large deviation estimate (5.33). The proof combines the ideas in Propositions 5.20 and 5.22. However here we construct a path g which is different from the one in the proof of Proposition 5.20 to ensure that the process $X(\cdot;x)$ returns to a neighbourhood of -1. The same proof implies exponential moments of the transition time $\tau_{B(\alpha;\delta)}(X(\cdot;x))$, but it is enough to estimate the second moment for the proof of Theorem 5.31.

Proposition 5.38. *Let* $\alpha \in (0, \alpha_0)$ *and* $\delta \in (0, 1/2)$. *For every* $\eta > 0$ *there exists* $\varepsilon_0 \in (0, 1)$ *such that for every* $\varepsilon \leq \varepsilon_0$

$$\sup_{x \in \mathcal{C}^{-\alpha_0}} \mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;x))^2 \leq C\mathrm{e}^{2[(V(0)-V(-1))+\eta]/\varepsilon}$$

for some $C > 0$ *independent of* η *and* ε.

Proof. We first prove that for every $R, \eta > 0$ there exists $T_0 > 0$ and $\varepsilon_0 \in (0, 1)$ such that for every $\varepsilon \leq \varepsilon_0$

$$\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}} \leq R} \mathbb{P}(\tau_{B(\alpha;\delta)}(X(\cdot;x)) \geq T_0) \leq 1 - \mathrm{e}^{-[(V(0)-V(-1))+\eta]/\varepsilon}.$$

We notice that there exists $\delta' > 0$ such that

$$\mathbb{P}(\tau_{B(\alpha;\delta)}(X(\cdot;x)) \leq T_0) \geq \mathbb{P}(\underbrace{\|X(T_*;x) - 1\|_{\mathcal{C}^{-\alpha}} \leq \delta' \text{ for some } T_* \leq T_0}_{=:\mathcal{A}(T_0;x)}).$$

Here we use that if $\|X(T_*;x)-1\|_{\mathcal{C}^{-\alpha}} \leq \delta'$, for δ' sufficiently small then $X(T_*;x) \in B(\alpha;\delta)$. By the large deviation estimate (5.33) we need to bound

$$\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}} \leq R} \sup_{\substack{f \in \mathcal{A}(T_0;x) \\ f(0)=x}} I(f(\cdot;x)).$$

To do so we proceed as in the proof of Proposition 5.20 by constructing a suitable path $g \in \mathcal{A}(T_0;x)$. The construction here is similar but some of the steps differ since we need to ensure that g returns to a neighbourhood of 1. To avoid repeating ourselves we give a sketch of the proof highlighting the different steps of the construction.

Steps 1, 2 and 3 are exactly as in the proof of Proposition 5.20. However we need to distinguish the value of δ there from the value of δ in the statement of the proposition. If $g(\tau_3;x) \in B_{\mathcal{B}^1_{2,2}}(1;\delta) \cap B_{\mathcal{C}^{2+\lambda}}(0;C)$ we stop at Step 3. If not then $g(\tau_3;x) \in B_{\mathcal{B}^1_{2,2}}(-1;\delta) \cap B_{\mathcal{C}^{2+\lambda}}(0;C)$ or $B_{\mathcal{B}^1_{2,2}}(0;\delta) \cap B_{\mathcal{C}^{2+\lambda}}(0;C)$. We only explain how to proceed in the first case since it also covers the other.

Before we describe the remaining steps we recall that by Proposition G.2 there exist $y_{0,-}, y_{0,+} \in B_{\mathcal{B}^1_{2,2}}(0;\delta)$ such that $y_{0,-}, y_{0,+} \in \mathcal{C}^\infty$ and $X_{det}(t;y_{0,\pm}) \to \pm 1$ in $\mathcal{B}^1_{2,2}$. In particular there exists $T_0^* > 0$ such that $X_{det}(T_0^*;y_{0,\pm}) \in B_{\mathcal{B}^1_{2,2}}(\pm 1;\delta) \cap B_{\mathcal{C}^{2+\lambda}}(0;C)$.

Step 4 (Jump to $X_{det}(T_0^*;y_{0,-})$):

Let $\tau_4 = \tau_3 + \tau$, for $\tau > 0$ as in Step 2 which we fix below according to Lemma 5.21. For $t \in [\tau_3, \tau_4]$ we set $g(t;x) = g(\tau_3;x) + \frac{t-\tau_3}{\tau_4-\tau_3}(X_{det}(T_0^*;y_{0,-}) - g(\tau_3;x))$.

Step 5 (Follow the deterministic flow backward to reach 0):

Let $\tau_5 = \tau_4 + T_0^*$. For $t \in [\tau_4, \tau_5]$ we set $g(t;x) = X_{det}(\tau_5 - t; y_{0,-})$.

Step 6 (Jump to $y_{0,+}$):

Let $\tau_6 = \tau_5 + \tau$, for τ as in Step 4. For $t \in [\tau_5, \tau_6]$ we set $g(t;x) = g(\tau_5;x) + \frac{t-\tau_5}{\tau_6-\tau_5}(y_{0,+} - g(\tau_5;x))$.

Step 7 (Follow the deterministic flow forward to reach 1):

Let $\tau_7 = \tau_6 + T_0^*$. For $t \in [\tau_6, \tau_7]$ we set $g(t;x) = X_{det}(t-\tau_6; y_{0,+})$.

For the path g constructed above we notice that for every $\|x\|_{\mathcal{C}^{-\alpha_0}} \le R$, if $t \ge \tau_7$ then $g(t;x) \in B_{\mathcal{B}^1_{2,2}}(1;\delta)$. By (A.6), $\mathcal{B}^1_{2,2} \subset \mathcal{C}^{-\alpha}$, for every $\alpha > 0$, hence if we choose δ sufficiently small and set $T_0 = \tau_7 + 1$ then $g \in \mathcal{A}(T_0;x)$.

To bound $I(g(\cdot;x))$ we proceed exactly as in the proof of Proposition 5.20 using Lemma 5.21. But when considering the contribution from Step 5 we get

$$\begin{aligned}
&\frac{1}{4}\int_{\tau_4}^{\tau_5} \|(\partial_t - \Delta)g(t;x) + g(t;x)^3 - g(t;x)\|_{L^2}^2 \,\mathrm{d}t \\
&\quad = 2\int_0^{T_0^*} \left\langle \partial_t X_{det}(t;y_{0,+}), \Delta X_{det}(t;y_{0,+}) - X_{det}(t;y_{0,+})^3 + X_{det}(t;y_{0,+})\right\rangle \mathrm{d}t \\
&\quad = -2\left(V(X_{det}(T_0^*;y_0)) - V(y_{0,+})\right) \\
&\quad \le 2\left(V(0) - V(-1)\right).
\end{aligned}$$

In total we obtain the bound

$$\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}} \le R} I(g(\cdot;x)) \le 2\left(V(0) - V(-1)\right) + C\delta.$$

For $\eta > 0$ we choose δ even smaller to ensure that $C\delta < \eta$. Then by (5.33) we find $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \le \varepsilon_0$

$$\inf_{\|x\|_{\mathcal{C}^{-\alpha_0}} \le R} \mathbb{P}(\tau_{B(\alpha;\delta)}(X(\cdot;x)) \le T_0) \ge \mathrm{e}^{-[(V(0)-V(-1))+\eta]/\varepsilon}.$$

The next step is to use the this estimate to show that for any $\eta > 0$ there exists $\varepsilon_0 \in (0,1)$ and possibly a different $T_0 > 0$ such that for every $\varepsilon \le \varepsilon_0$

$$\sup_{x\in\mathcal{C}^{-\alpha_0}} \mathbb{P}(\tau_{B(\alpha;\delta)}(X(\cdot;x)) \ge mT_0) \le \left(1 - \mathrm{e}^{-[(V(0)-V(-1))+\eta]/\varepsilon}\right)^m .$$

We omit the proof since it is the same as the one of Proposition 5.22.

Finally we notice that

$$\begin{aligned}
\mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;x))^2 &= \int_0^\infty 2t\,\mathbb{P}(\tau_{B(\alpha;\delta)}(X(\cdot;x)) \ge t)\,\mathrm{d}t \\
&\le \sum_{m=0}^\infty \mathbb{P}(\tau_{B(\alpha;\delta)}(X(\cdot;x)) \ge mT_0) \int_{mT_0}^{(m+1)T_0} 2t\,\mathrm{d}t \\
&\le 2T_0^2 \sum_{m=0}^\infty (m+1)\left(1 - \mathrm{e}^{-[(V(0)-V(-1))+\eta]/\varepsilon}\right)^m \\
&= 2T_0^2 \mathrm{e}^{2[(V(0)-V(-1))+\eta]/\varepsilon}
\end{aligned}$$

which completes the proof. $\square$

Proof of Theorem 5.31. Let

$$\mathtt{Pr}(\varepsilon) = \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k} \exp\left\{\frac{\nu_k - \lambda_k}{\lambda_k + 2}\right\}}\mathrm{e}^{(V(0)-V(-1))/\varepsilon}$$

and $\delta \in (0, \delta_0)$, for $\delta_0 \in (0, 1/2)$ which we fix below.

To prove the upper bound in (5.53) let $\delta_- < \delta$ and $T > 0$ which we also fix below. For $x \in A(\alpha_0; \delta_-)$ we define the set

$$\begin{aligned}
&A_T(x) \\
&\quad = \Bigg\{\tau_{B(\alpha;\delta_-)}(X(\cdot;x)) > T, \sup_{\|\bar{y}-x\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0} \frac{\|X(t;\bar{y}) - X(t;x)\|_{\mathcal{C}^\beta}}{\|\bar{y} - x\|_{\mathcal{C}^{-\alpha_0}}} \le C\mathrm{e}^{-(2-\kappa)t} \\
&\qquad\quad \text{for every } t \ge T\Bigg\}
\end{aligned}$$

where δ_0 and C are as in Theorem 5.3. For $y \in A(\alpha_0; \delta)$ and $x \in A(\alpha_0; \delta_-)$ we have that $\|y - x\|_{\mathcal{C}^{-\alpha_0}}, \|x - (-1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0$, if we choose δ_0 sufficiently small. Furthermore for $y \in A(\alpha_0; \delta)$, $x \in A(\alpha_0; \delta_-)$ and $\omega \in A_T(x)$

$$\tau_{B(\alpha;\delta)}(X(\cdot;y)) \le \tau_{B(\alpha;\delta_-)}(X(\cdot;x)),$$

if we choose T sufficiently large. By Proposition 5.37 and Theorem 5.3 there exist $a_1 > 0$ and $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \le \varepsilon_0$

$$\sup_{x \in A(\alpha_0;\delta_-)} \mathbb{P}(A_T(x)^c) \le \sup_{\|x-(-1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0} \mathbb{P}(A_T(x)^c) \le \mathrm{e}^{-a_1/\varepsilon}.$$

Then for every $y \in A(\alpha_0;\delta)$, $x \in A(\alpha_0;\delta_-)$ and $\eta > 0$, which we fix below, there exists $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \le \varepsilon_0$

$$\begin{aligned}
&\mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;y)) \\
&\quad \le \mathbb{E}\tau_{B(\alpha;\delta_-)}(X(\cdot;x)) + \mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;y))\mathbf{1}_{A_T(x)^c} \\
&\quad \overset{\text{Cauchy-Schwarz}}{\le} \mathbb{E}\tau_{B(\alpha;\delta_-)}(X(\cdot;x)) + \left(\mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;y))^2\right)^{\frac{1}{2}} \mathbb{P}(A_T(x)^c)^{\frac{1}{2}} \\
&\quad \overset{\text{Prop. 5.38}}{\le} \mathbb{E}\tau_{B(\alpha;\delta_-)}(X(\cdot;x)) + C\mathrm{e}^{\left((V(0)-V(-1))+\eta-\frac{a_1}{2}\right)/\varepsilon}
\end{aligned} \tag{5.64}$$

for some $C > 0$ independent of ε. By Proposition 5.36 there exist $\delta_- \in (0,\delta)$, $c_+ > 0$ and $\mu_\varepsilon \in \mathcal{M}_1(A(\alpha_0;\delta_-))$ such that for every $\varepsilon \in (0,1)$

$$\int \mathbb{E}\tau_{B(\alpha;\delta_-)}(X(\cdot;x))\,\mu_\varepsilon(\,\mathrm{d}x) \le \mathtt{Pr}(\varepsilon)(1+c_+\sqrt{\varepsilon}).$$

Integrating (5.64) over x with respect to μ_ε implies that

$$\begin{aligned}
&\sup_{y \in A(\alpha_0;\delta)} \mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;y)) \le \mathtt{Pr}(\varepsilon) \\
&\quad \times \left((1+c_+\sqrt{\varepsilon}) + \mathrm{e}^{\left(\eta-\frac{a_1}{2}\right)/\varepsilon} C \left(\frac{2\pi}{|\lambda_0|} \sqrt{\prod_{k \in \mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k} \exp\left\{ \frac{\nu_k - \lambda_k}{\lambda_k + 2} \right\}} \right)^{-1} \right).
\end{aligned}$$

Let $\zeta > 0$. Choosing $\eta < \frac{a_1}{2}$ we can find $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \le \varepsilon_0$

$$\mathrm{e}^{\left(\eta-\frac{a_1}{2}\right)/\varepsilon} C \left(\frac{2\pi}{|\lambda_0|} \sqrt{\prod_{k \in \mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k} \exp\left\{ \frac{\nu_k - \lambda_k}{\lambda_k + 2} \right\}} \right)^{-1} \le \zeta\sqrt{\varepsilon}$$

which in turn implies that

$$\sup_{y \in A(\alpha_0;\delta)} \mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;y)) \le \mathtt{Pr}(\varepsilon)\left(1+(c_+ + \zeta)\sqrt{\varepsilon}\right)$$

and proves the upper bound in (5.53).

To prove the lower bound, we let $\delta_+ \in (\delta, \delta_0)$ which we fix below and for $y \in A(\alpha_0; \delta)$ and $x \in A(\alpha_0; \delta_+)$ we define the set

$$\begin{aligned} & B_T(y,x) \\ & \quad = \Bigg\{ \tau_{B(\alpha;\delta)}(X(\cdot;y)) \geq T, \sup_{\|\bar{y}-x\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0} \frac{\|X(t;\bar{y}) - X(t;x)\|_{\mathcal{C}^\beta}}{\|\bar{y}-x\|_{\mathcal{C}^{-\alpha_0}}} \leq C\mathrm{e}^{-(2-\kappa)t} \\ & \qquad \text{for every } t \geq T \Bigg\}. \end{aligned}$$

For $y \in A(\alpha_0; \delta)$ and $x \in A(\alpha_0; \delta_+)$ we have that $\|y-x\|_{\mathcal{C}^{-\alpha_0}}, \|y-(-1)\|_{\mathcal{C}^{-\alpha_0}}, \|x-(-1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0$, if we choose δ_0 sufficiently small. We also notice that for $y \in A(\alpha_0; \delta)$, $x \in A(\alpha_0; \delta_+)$ and $\omega \in B_T(y,x)$

$$\tau_{B(\alpha;\delta_+)}(X(\cdot;x)) \leq \tau_{B(\alpha;\delta)}(X(\cdot;y)),$$

if we choose T sufficiently large. By Proposition 5.37 and Theorem 5.3 there exists $a_1 > 0$ and $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \leq \varepsilon_0$

$$\sup_{\substack{y \in A(\alpha_0;\delta) \\ x \in A(\alpha_0;\delta_+)}} \mathbb{P}(B_T(y,x)^c) \leq \sup_{\substack{\|y-(-1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0 \\ \|x-(-1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0}} \mathbb{P}(B_T(y,x)^c) \leq \mathrm{e}^{-a_1/\varepsilon}.$$

Then for every $y \in A(\alpha_0; \delta)$, $x \in A(\alpha_0; \delta_+)$ and $\varepsilon \leq \varepsilon_0$

$$\begin{aligned} & \mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;y)) \\ & \quad \geq \mathbb{E}\tau_{B(\alpha;\delta_+)}(X(\cdot;x)) \mathbf{1}_{B_T(y,x)} \\ & \quad = \mathbb{E}\tau_{B(\alpha;\delta_+)}(X(\cdot;x)) - \mathbb{E}\tau_{B(\alpha;\delta_+)}(X(\cdot;x)) \mathbf{1}_{B_T(y,x)^c} \\ & \quad \overset{\text{Cauchy-Schwarz}}{\geq} \mathbb{E}\tau_{B(\alpha;\delta_+)}(X(\cdot;x)) - \left(\mathbb{E}\tau_{B(\alpha;\delta_+)}(X(\cdot;x))^2\right)^{\frac{1}{2}} \mathbb{P}\left(B_T(y,x)^c\right)^{\frac{1}{2}} \\ & \quad \geq \mathbb{E}\tau_{B(\alpha;\delta_+)}(X(\cdot;x)) - \left(\mathbb{E}\tau_{B(\alpha;\delta_+)}(X(\cdot;x))^2\right)^{\frac{1}{2}} \mathrm{e}^{-a_1/2\varepsilon} \end{aligned}$$

and we proceed as in the case of the upper bound, using Proposition 5.38 for $\mathbb{E}\tau_{B(\alpha;\delta_+)}(X(\cdot;x))^2$ and Proposition 5.36 to find $\delta_+ \in (\delta, \delta_0)$, $c_- > 0$ and $\mu_\varepsilon \in \mathcal{M}_1\left(A(\alpha_0; \delta_+)\right)$ such that for every $\varepsilon \in (0,1)$

$$\int \mathbb{E}\tau_{B(\alpha;\delta_+)}(X(\cdot;x))\, \mu_\varepsilon(\mathrm{d}x) \geq \mathtt{Pr}(\varepsilon)(1 - c_-\varepsilon).$$

$\square$

Appendix

Appendix A

In this appendix we present several useful results from [MW17c, MW17b] about Besov spaces. For a complete survey of the full-space analogues of these results we refer the reader to [BCD11]. A discussion on the validity of these results in the periodic case can be found in [MW17c, Section 4.2].

The following estimates are immediate from the definition of the Besov norm (1.13).

$$\|f\|_{\mathcal{B}^{\alpha_1}_{p_1,q_1}} \leq C\|f\|_{\mathcal{B}^{\alpha_2}_{p_1,q_1}}, \text{ whenever } \alpha_1 < \alpha_2. \tag{A.1}$$

$$\|f\|_{\mathcal{B}^{\alpha_1}_{p_1,q_1}} \leq \|f\|_{\mathcal{B}^{\alpha_1}_{p_1,q_2}}, \text{ whenever } q_1 > q_2. \tag{A.2}$$

$$\|f\|_{\mathcal{B}^{\alpha_1}_{p_1,q_1}} \leq C\|f\|_{\mathcal{B}^{\alpha_1}_{p_2,q_1}}, \text{ whenever } p_1 < p_2. \tag{A.3}$$

Proposition A.1 ([MW17c, Remark 9])**.** *Let* $p, q_1, q_2 \in [1, \infty]$ *such that* $q_2 > q_1$. *For every* $\alpha_2 > \alpha_1$

$$\|f\|_{\mathcal{B}^{\alpha_1}_{p_1,q_1}} \leq C\|f\|_{\mathcal{B}^{\alpha_2}_{p_1,q_2}}. \tag{A.4}$$

Proposition A.2 ([MW17c, Remarks 10 and 11])**.** *For every* $p \in [1, \infty]$

$$C^{-1}\|f\|_{\mathcal{B}^{0}_{p,\infty}} \leq \|f\|_{L^p} \leq C\|f\|_{\mathcal{B}^{0}_{p,1}}. \tag{A.5}$$

Proposition A.3 ([MW17c, Proposition 2])**.** *Let* $\beta \geq \alpha$ *and* $p, q \geq 1$ *such that* $p \geq q$ *and* $\beta = \alpha + d\left(\frac{1}{q} - \frac{1}{p}\right)$. *Then*

$$\|f\|_{\mathcal{B}^{\alpha}_{p,\infty}} \leq C\|f\|_{\mathcal{B}^{\beta}_{q,\infty}}. \tag{A.6}$$

Proposition A.4 ([MW17c, Proposition 10])**.** *For every* $\alpha < \alpha'$ *the embedding* $\mathcal{C}^{\alpha'} \hookrightarrow \mathcal{B}^{\alpha}_{\infty,1}$ *is compact.*

In the following proposition we describe the smoothing properties of the heat semigroup $\left(\mathrm{e}^{t\Delta}\right)_{t\geq 0}$ with generator Δ in space.

Proposition A.5 ([MW17c, Proposition 5]). *For every $\beta \geq \alpha$*

$$\|\mathrm{e}^{t\Delta} f\|_{\mathcal{B}^{\beta}_{p,q}} \leq C(t \wedge 1)^{\frac{\alpha-\beta}{2}} \|f\|_{\mathcal{B}^{\alpha}_{p,q}}. \tag{A.7}$$

For $f, g \in \mathcal{C}^{\infty}$ we define the paraproduct $f \prec g$ and the resonant term $f \circ g$ by

$$f \prec g := \sum_{\iota < \kappa - 1} \delta_{\iota} f \delta_{\kappa} g, \tag{A.8}$$

$$f \circ g := \sum_{|\iota - \kappa| \leq 1} \delta_{\iota} f \delta_{\kappa} g. \tag{A.9}$$

We also let $f \succ g := g \prec f$. Notice that formally

$$fg = f \prec g + f \circ g + f \succ g.$$

We then have the following estimates due to Bony.

Proposition A.6 ([BCD11, Theorems 2.82 and 2.85]). *Let $\alpha, \beta \in \mathbb{R}$ and $g \in \mathcal{C}^{\beta}$.*

i. If $f \in L^{\infty}$, $\|f \prec g\|_{\mathcal{C}^{\beta}} \leq C\|f\|_{L^{\infty}}\|g\|_{\mathcal{C}^{\beta}}$.

ii. If $\alpha < 0$ and $f \in \mathcal{C}^{\alpha}$, $\|f \prec g\|_{\mathcal{C}^{\alpha+\beta}} \leq C\|f\|_{\mathcal{C}^{\alpha}}\|g\|_{\mathcal{C}^{\beta}}$.

iii. If $\alpha + \beta > 0$ and $f \in \mathcal{C}^{\alpha}$, $\|f \circ g\|_{\mathcal{C}^{\alpha+\beta}} \leq C\|f\|_{\mathcal{C}^{\alpha}}\|g\|_{\mathcal{C}^{\beta}}$.

We have the following two propositions for products of distributions in Besov spaces.

Proposition A.7 ([MW17c, Corollary 1]). *Let $\alpha \geq 0$ and $p, q \in [1, \infty]$. Then*

$$\|fg\|_{\mathcal{B}^{\alpha}_{p,q}} \leq C\|f\|_{\mathcal{B}^{\alpha}_{p_1,q_1}}\|g\|_{\mathcal{B}^{\alpha}_{p_2,q_2}}, \tag{A.10}$$

where $p = \frac{1}{p_1} + \frac{1}{p_2}$ and $p = \frac{1}{q_1} + \frac{1}{q_2}$.

Proposition A.8 ([MW17c, Corollary 2]). *Let $\alpha < 0$, $\beta > 0$ such that $\alpha + \beta > 0$ and $p, q \in [1, \infty]$. Then*

$$\|fg\|_{\mathcal{B}^{\alpha}_{p,q}} \leq C\|f\|_{\mathcal{B}^{\alpha}_{p_1,q_1}}\|g\|_{\mathcal{B}^{\beta}_{p_2,q_2}}, \tag{A.11}$$

where $p = \frac{1}{p_1} + \frac{1}{p_2}$ and $p = \frac{1}{q_1} + \frac{1}{q_2}$.

The following is an extension of the L^2-inner product in Besov spaces.

Proposition A.9 ([MW17c, Proposition 7]). *Let* $\alpha \in [0,1)$ *and* $p,q,p',q' \in [1,\infty]$ *such that* $\frac{1}{p}+\frac{1}{p'}=1$ *and* $\frac{1}{q}+\frac{1}{q'}=1$. *Then*

$$|\langle f,g\rangle| \leq C\|f\|_{\mathcal{B}^{\alpha}_{p,q}}\|g\|_{\mathcal{B}^{-\alpha}_{p',q'}}. \tag{A.12}$$

We have the following gradient estimates for functions of positive regularity.

Proposition A.10 ([MW17c, Proposition 8]). *For every* $\alpha \in (0,1)$

$$\|f\|_{\mathcal{B}^{\alpha}_{1,1}} \leq C\left(\|f\|_{L^1}^{1-\alpha}\|\nabla f\|_{L^1}^{\alpha}+\|f\|_{L^1}\right). \tag{A.13}$$

Proposition A.11 ([MW17b, Proposition A.6]). *For every* $p \in [1,\infty)$

$$\|f\|_{\mathcal{B}^{1}_{p,\infty}} \leq C(\|\nabla f\|_{L^p}+\|f\|_{L^p}).$$

Proposition A.12 ([MW17b, Corollary A.8]). *Let* $\alpha>0$ *and* $p,q\in[1,\infty]$. *Then*

$$\|f^2\|_{\mathcal{B}^{\alpha}_{p,q}} \leq C\|f\|_{L^{p_1}}\|f\|_{\mathcal{B}^{\alpha}_{p_2,q}}, \tag{A.14}$$

where $p=\frac{1}{p_1}+\frac{1}{p_2}$.

In the next proposition we prove convergence of the Galerkin approximations $\Pi_N f$ to f in Besov spaces. Here we use that the projection $\Pi_N f$ is defined as the convolution of f with the 2-dimensional square Dirichlet kernel, which satisfies a logarithmic growth bound in the L^1 norm.

Proposition A.13. *Let* $\Pi_N : L^2 \to L^2$ *be the projection on* $\{f \in L^2 : f(z) = \sum_{|k|\leq N}\hat{f}(k)L^{-2}\mathrm{e}^{2\mathrm{i}\pi k\cdot z/L}\}$. *Then for every* $\alpha\in\mathbb{R}$, $p,q\in[1,\infty]$ *and* $\lambda>0$

$$\|\Pi_N f - f\|_{\mathcal{B}^{\alpha}_{p,q}} \leq \frac{C(\log N)^2}{N^{\lambda}}\|f\|_{\mathcal{B}^{\alpha+\lambda}_{p,q}} \tag{A.15}$$

$$\|\Pi_N f\|_{\mathcal{B}^{\alpha}_{p,q}} \leq C\|f\|_{\mathcal{B}^{\alpha+\lambda}_{p,q}}. \tag{A.16}$$

If we furthermore assume that $p=2$ *then*

$$\|\Pi_N f - f\|_{\mathcal{B}^{\alpha}_{2,q}} \leq \frac{C}{N^{\lambda}}\|f\|_{\mathcal{B}^{\alpha+\lambda}_{2,q}} \tag{A.17}$$

$$\|\Pi_N f\|_{\mathcal{B}^{\alpha}_{2,q}} \leq \|f\|_{\mathcal{B}^{\alpha}_{2,q}}. \tag{A.18}$$

Proof. We first notice that for $c_2 > c_1 > 0$

$$\delta_\kappa\left(\Pi_N f - f\right) = \begin{cases} 0 & , \text{ if } 2^\kappa \le c_1 N \\ \delta_\kappa f & , \text{ if } 2^\kappa > c_2 N \end{cases}.$$

Let $D_N(z) = \sum_{|k|\le N} L^{-2}\mathrm{e}^{-2\mathrm{i}\pi k\cdot z/L}$ be the square Dirichlet kernel. Then $\Pi_N f = f * D_N$. Using the triangle inequality and Young's inequality for convolution we have that

$$\|\delta_\kappa\left(\Pi_N f - f\right)\|_{L^p} \le (\|D_N\|_{L^1} + 1)\|\delta_\kappa f\|_{L^p}.$$

Thus

$$\|\delta_\kappa(\Pi_N f - f)\|_{L^p} \le \begin{cases} 0 & , \text{ if } 2^\kappa \le c_1 N \\ C(\log N)^2\|\delta_\kappa f\|_{L^p} & , \text{ if } c_1 N \le 2^\kappa < c_2 N \\ \|\delta_\kappa f\|_{L^p} & , \text{ if } 2^\kappa > c_2 N \end{cases}$$

where in the second case we use that $\|D_N\|_{L^1} \lesssim (\log N)^2$. This bound immediate form the fact that the 2-dimensional square Dirichlet kernel is the product of two 1-dimensional Dirichlet kernels (see [Gra14, Section 3.1.3]). The last implies (A.15) and (A.16). For $p = 2$ we notice that

$$\|\delta_\kappa \Pi_N f\|_{L^2} \le \|\delta_\kappa f\|_{L^2}$$

which implies (A.17) and (A.18). □

Appendix B

Definition B.1. Let $\{\xi(\phi)\}_{\phi\in L^2(\mathbb{R}\times\mathbb{T}^d)}$ be a family of centered Gaussian random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\mathbb{E}(\xi(\phi)\xi(\psi)) = \langle\phi, \psi\rangle_{L^2(\mathbb{R}\times\mathbb{T}^d)},$$

for all $\phi, \psi \in L^2(\mathbb{R}\times\mathbb{T}^d)$. Then ξ is called a space-time white noise on $\mathbb{R}\times\mathbb{T}^d$.

The existence of such a family of random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is assured by Kolmogorov's extension theorem and by definition we can check that it is linear, i.e. for all $\lambda, \nu \in \mathbb{R}$, $\phi, \psi \in L^2(\mathbb{R}\times\mathbb{T}^d)$ we have that $\xi(\lambda\phi+\nu\psi)=\lambda\xi(\phi)+\nu\xi(\psi)$ $\mathbb{P}$-almost surely (see [Nua06, Chapter 1]). We interpret $\xi(\phi)$ as a stochastic integral and write

$$\int_{\mathbb{R}\times\mathbb{T}^d}\phi(t,x)\xi(\,\mathrm{d}t,\,\mathrm{d}x):=\xi(\phi),$$

for all $\phi\in L^2(\mathbb{R}\times\mathbb{T}^d)$. We use this notation, but stress that ξ is almost surely not a measure and that the stochastic integral is only defined on a set of measure one which my depend on the specific choice of ϕ.

We also define multiple stochastic integrals (see [Nua06, Chapter 1]) on $\mathbb{R}\times\mathbb{T}^d$ for all symmetric functions f in $L^2\left((\mathbb{R}\times\mathbb{T}^d)^n\right)$, for some $n\in\mathbb{N}$, i.e. functions such that $f(z_1,z_2,\ldots,z_n)=f(z_{i_1},z_{i_2},\ldots,z_{i_n})$ for any permutation $(i_1,i_2,\ldots,i_n)$ of $(1,2,\ldots,n)$. Here z_j is an element of $\mathbb{R}\times\mathbb{T}^d$, for all $j\in\{1,2,\ldots,n\}$. For such a symmetric function f we denote its n-th iterated stochastic integral by

$$I_n(f):=\int_{(\mathbb{R}\times\mathbb{T}^d)^n}f(z_1,z_2,\ldots,z_n)\,\xi(\,\mathrm{d}z_1\otimes\,\mathrm{d}z_2\otimes\ldots\otimes\,\mathrm{d}z_n).$$

The following theorem can be found in [Nua06, Theorem 1.1.2].

Theorem B.2. *Let $\mathcal{F}_\xi$ be the σ-algebra generated by the family $\{\xi(\phi)\}_{\phi\in L^2(\mathbb{R}\times\mathbb{T}^d)}$. Then every element $X\in L^2(\Omega,\mathcal{F}_\xi,\mathbb{P})$ can be written in the following form*

$$X=\mathbb{E}(X)+\sum_{n=1}^{\infty}I_n(f_n),$$

where $f_n\in L^2\left((\mathbb{R}\times\mathbb{T}^d)^n\right)$ are symmetric functions, uniquely determined by X.

This theorem implies that $L^2(\Omega,\mathcal{F}_\xi,\mathbb{P})$ can be decomposed into the direct sum $\bigoplus_{n\geq 0}S_n$, where $S_0:=\mathbb{R}$ and

$$S_n:=\{I_n(f):f\in L^2\left((\mathbb{R}\times\mathbb{T}^d)^n\right)\text{ symmetric}\},\tag{B.1}$$

for all $n\geq 1$. The space S_n is called the n-th homogeneous Wiener chaos and the element $I_n(f_n)$ the projection of X onto S_n.

Given a symmetric function $f\in L^2\left((\mathbb{R}\times\mathbb{T}^d)^n\right)$, we have the isometry

$$\mathbb{E}(I_n(f))^2=n!\|f\|^2_{L^2\left((\mathbb{R}\times\mathbb{T}^d)^n\right)}.\tag{B.2}$$

Furthermore, by Nelson's estimate (see [Nua06, Section 1.4]) for every $n\geq 1$ and $Y\in S_n$,

$$\mathbb{E}|Y|^p\leq(p-1)^{\frac{n}{2}p}\left(\mathbb{E}|Y|^2\right)^{\frac{p}{2}},\tag{B.3}$$

for every $p\geq 2$.

Appendix C

Definition C.1. For symmetric kernels $K_1, K_2 : \mathbb{Z}^2 \to (0,\infty)$ we denote by $K_1 \star K_2$ the convolution given by

$$K_1 \star K_2(m) := \sum_{l \in \mathbb{Z}^2} K_1(m-l) K_2(l)$$

and for $N \in \mathbb{N}$ we let

$$K_1 \star_{\leq N} K_2(m) := \sum_{|l| \leq N} K_1(m-l) K_2(l).$$

as well as

$$K_1 \star_{>N} K_2 := (K_1 \star K_2) - (K_1 \star_{\leq N} K_2).$$

We are interested in symmetric kernels K for which there exists $\alpha \in (0,1]$ such that

$$K(m) \leq C \frac{1}{(1+|m|^2)^\alpha}.$$

In the spirit of [Hai14, Lemma 10.14] we have the following lemma.

Lemma C.2. *Let $\alpha, \beta \in (0,1]$ such that $\alpha+\beta-1>0$ and let $K_1, K_2 : \mathbb{Z}^2 \to (0,\infty)$ be symmetric kernels such that*

$$K_1(m) \leq C \frac{1}{(1+|m|^2)^\alpha}, \quad K_2(m) \leq C \frac{1}{(1+|m|^2)^\beta}.$$

If $\alpha < 1$ or $\beta < 1$ then

$$K_1 \star K_2(m) \leq C \frac{1}{(1+|m|^2)^{\alpha+\beta-1}}$$

$$K_1 \star_{>N} K_2(m) \leq C \begin{cases} \frac{1}{(1+|m|^2)^{\alpha+\beta-1}}, & \textit{if } |m| \geq N \\ \frac{1}{(1+|N|^2)^{\alpha+\beta-1}}, & \textit{if } |m| < N \end{cases}$$

and if $\alpha = \beta = 1$

$$K_1 \star K_2(m) \leq C \frac{\log|m| \vee 1}{1+|m|^2}$$

$$K_1 \star_{>N} K_2(m) \leq C \begin{cases} \frac{\log|m| \vee 1}{1+|m|^2}, & \textit{if } |m| \geq N \\ \frac{\log|N| \vee 1}{1+|N|^2}, & \textit{if } |m| < N \end{cases}.$$

Proof. We only prove the estimates for $K_1 \star K_2$. The corresponding estimates for $K_1 \star_{>N} K_2$ can be proven in a similar way. We consider the following regions of $\mathbb{Z}^2$,

$$\begin{aligned}
A_1 &= \left\{ l : |l| \le \frac{|m|}{2} \right\}, \\
A_2 &= \left\{ l : |l-m| \le \frac{|m|}{2} \right\}, \\
A_3 &= \left\{ l : \frac{|m|}{2} \le |l| \le 2|m|, |l-m| \ge \frac{|m|}{2} \right\}, \\
A_4 &= \{ l : |l| > 2|m| \}.
\end{aligned}$$

For every $l \in A_1$ we notice that $|m-l| \ge \frac{3|m|}{4}$, which implies that

$$\begin{aligned}
\sum_{l \in A_1} K_1(m-l) K_2(l) &\lesssim \frac{1}{(1+|m|^2)^\alpha} \sum_{l \in A_1} K_2(l) \\
&\lesssim \begin{cases} \frac{(1+|m|^2)^{\beta-1}}{(1+|m|^2)^\alpha}, & \text{if } \beta < 1 \\ \frac{\log|m| \vee 1}{(1+|m|^2)^\alpha}, & \text{if } \beta = 1 \end{cases}.
\end{aligned}$$

By symmetry we get that

$$\sum_{l \in A_2} K_1(m-l) K_2(l) \lesssim \begin{cases} \frac{(1+|m|^2)^{\alpha-1}}{(1+|m|^2)^\beta}, & \text{if } \alpha < 1 \\ \frac{\log|m| \vee 1}{(1+|m|^2)^\beta}, & \text{if } \alpha = 1 \end{cases}.$$

For the summation over A_3 we notice that

$$\sum_{l \in A_3} K_1(m-l) K_2(l) \lesssim \frac{1+|m|^2}{(1+|m|^2)^{\alpha+\beta}}.$$

Finally, for $l \in A_4$ we have that $|m-l| \ge \frac{|l|}{2}$, which implies that

$$\sum_{l \in A_4} K_1(m-l) K_2(l) \lesssim \sum_{|l|>2|m|} \frac{1}{(1+|l|^2)^{\alpha+\beta}} \lesssim \frac{1}{(1+|m|^2)^{\alpha+\beta}}.$$

Combining all the above we thus obtain the appropriate estimate on $K_1 \star K_2(m)$. $\square$

Because we are interested in nested convolutions of the same kernel we introduce the following recursive notation

$$K \star^1 K = K, \quad K \star^n K = K \star \left(K \star^{n-1} K \right),$$

for every $n \geq 2$, with the obvious interpretation for $K \star^n_{\leq N} K$ and $K \star^n_{>N} K$. We then have the following corollary, the proof of which is omitted since it is an immediate consequence of Lemma C.2.

Corollary C.3. *Let K be a symmetric kernel as above for some $\alpha \in (\frac{n-1}{n}, 1]$. If $\alpha < 1$ then*

$$K \star^n K(m) \leq C \frac{1}{(1+|m|^2)^{n\alpha-(n-1)}}$$

$$K \star^n_{>N} K(m) \leq C \begin{cases} \frac{1}{(1+|m|^2)^{n\alpha-(n-1)}}, & \text{if } |m| \geq N \\ \frac{1}{(1+|N|^2)^{n\alpha-(n-1)}}, & \text{if } |m| < N \end{cases}$$

and if $\alpha = 1$

$$K \star^n K(m) \leq C \frac{1}{(1+|m|^2)^{1-\varepsilon}}$$

$$K \star_{>N} K(m) \leq C \begin{cases} \frac{1}{(1+|m|^2)^{1-\varepsilon}}, & \text{if } |m| \geq N \\ \frac{1}{(1+|N|^2)^{1-\varepsilon}}, & \text{if } |m| < N \end{cases}$$

for every $\varepsilon \in (0,1)$.

Appendix D

Proof of Theorem 2.1. Let $\phi_1, \phi_2 \in L^2$ and notice that for $t_1, t_2 > -\infty$ by (B.2)

$$\begin{aligned} &\mathbb{E}\langle \stackrel{n}{\triangledown}_{-\infty}(t_1), \phi_1\rangle\langle \stackrel{n}{\triangledown}_{-\infty}(t_2), \phi_2\rangle \\ &\quad = n! \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \phi_1(z_1)\phi_2(z_2)\left(\int_{-\infty}^{t_1\wedge t_2} H(t_1+t_2-2r, z_1-z_2)\,\mathrm{d}r\right)^n \mathrm{d}z_1\,\mathrm{d}z_2, \end{aligned} \tag{D.1}$$

where we also use the semigroup property

$$\int_{\mathbb{T}^2} H(t_1-r, z_1-z)H(t_2-r, z_2-z)\,\mathrm{d}z = H(t_1+t_2-2r, z_1-z_2).$$

For $I_m = 1 + 4\pi^2|m|^2$, $m \in \mathbb{Z}^2$, we rewrite (D.1) as

$$\begin{aligned} &\mathbb{E}\langle \stackrel{n}{\triangledown}_{-\infty}(t_1), \phi_1\rangle\langle \stackrel{n}{\triangledown}_{-\infty}(t_2), \phi_2\rangle \\ &\quad = n! \sum_{\substack{m_i\in\mathbb{Z}^2 \\ i=1,2,\dots,n \\ m=m_1+\dots+m_n}} \prod_{i=1}^n \frac{\mathrm{e}^{-I_{m_i}|t_1-t_2|}}{2I_{m_i}} \hat{\phi}_1(m)\overline{\hat{\phi}_2(m)}, \end{aligned}$$

and if we replace ϕ_1, ϕ_2 by $\eta_\kappa(z_1-\cdot)$, $\eta_\kappa(z_2-\cdot)$ respectively, for $\kappa \geq -1$, $z_1, z_2 \in \mathbb{T}^2$, we have that

$$\begin{aligned}
&\mathbb{E}\left(\delta_\kappa \overset{n}{\triangledown}_{-\infty}(t_1)\right)(z_1)\left(\delta_\kappa \overset{n}{\triangledown}_{-\infty}(t_2)\right)(z_2) \\
&= n! \sum_{\substack{m_i\in\mathbb{Z}^2 \\ i=1,2,\ldots,n \\ m=m_1+\ldots+m_n}} \prod_{i=1}^{n} \frac{\mathrm{e}^{-I_{m_i}|t_1-t_2|}}{2I_{m_i}} |\chi_\kappa(m)|^2 e_m(z_1-z_2).
\end{aligned}$$

By a change of variables we finally obtain

$$\begin{aligned}
&\mathbb{E}\left(\delta_\kappa \overset{n}{\triangledown}_{-\infty}(t_1)\right)(z_1)\left(\delta_\kappa \overset{n}{\triangledown}_{-\infty}(t_2)\right)(z_2) \\
&\approx n! \sum_{m_1\in\mathcal{A}_{2^\kappa}} \sum_{\substack{m_i\in\mathbb{Z}^2 \\ i=2,\ldots,n}} \prod_{i=1}^{n} \frac{\mathrm{e}^{-I_{m_i-m_{i-1}}|t_1-t_2|}}{2I_{m_i-m_{i-1}}} e_{m_1}(z_1-z_2),
\end{aligned}$$

with the convention that $m_0 = 0$. Let $K^\gamma(m) = \frac{1}{(1+|m|^2)^{1-\gamma}}$, for $\gamma \in [0,1)$, and write $K^\gamma \star^n K^\gamma$ to denote the n-th iterated convolution of K^γ with itself (see Definition C.1). If we let $z_1 = z_2 = z$, for $t_1 = t_2 = t$ we get an estimate of the form

$$\mathbb{E}\left(\delta_\kappa \overset{n}{\triangledown}_{-\infty}(t)\right)(z)^2 \lesssim \sum_{m\in\mathcal{A}_{2^\kappa}} K^0 \star^n K^0(m)$$

while for $t_1 \neq t_2$ and every $\gamma \in (0,1)$

$$\mathbb{E}\left[\left(\delta_\kappa \overset{n}{\triangledown}_{-\infty}(t_1)\right)(z) - \left(\delta_\kappa \overset{n}{\triangledown}_{-\infty}(t_2)\right)(z)\right]^2 \lesssim |t_1-t_2|^{n\gamma} \sum_{m\in\mathcal{A}_{2^\kappa}} K^\gamma \star^n K^\gamma(m).$$

By Corollary C.3

$$\mathbb{E}\left(\delta_\kappa \overset{n}{\triangledown}_{-\infty}(t)\right)(z)^2 \lesssim \sum_{m\in\mathcal{A}_{2^\kappa}} \frac{1}{(1+|m|^2)^{1-\varepsilon}},$$

for every $\varepsilon \in (0,1)$, and

$$\mathbb{E}\left[\left(\delta_\kappa \overset{n}{\triangledown}_{-\infty}(t_1)\right)(z) - \left(\delta_\kappa \overset{n}{\triangledown}_{-\infty}(t_2)\right)(z)\right]^2 \lesssim |t_1-t_2|^{n\gamma} \sum_{m\in\mathcal{A}_{2^\kappa}} \frac{1}{(1+|m|^2)^{1-n\gamma}}.$$

Using the fact that $m \in \mathcal{A}_{2^\kappa}$ we have that for every $\kappa \geq -1$

$$\begin{aligned}
&\mathbb{E}\left(\delta_\kappa \overset{n}{\triangledown}_{-\infty}(t)\right)(z)^2 \lesssim 2^{2\lambda_1\kappa} \\
&\mathbb{E}\left[\left(\delta_\kappa \overset{n}{\triangledown}_{-\infty}(t_1)\right)(z) - \left(\delta_\kappa \overset{n}{\triangledown}_{-\infty}(t_2)\right)(z)\right]^2 \lesssim |t_1-t_2|^{n\gamma} 2^{2\lambda_2\kappa}
\end{aligned}$$

for every $\lambda_1 > 0$ and every $\gamma \in (0, \frac{1}{n})$, $\lambda_2 > n\gamma$, while for every $p \geq 2$ by Nelson's estimate (B.3) we finally get

$$\begin{aligned}
&\mathbb{E}\left(\delta_\kappa \overset{n}{\triangledown}_{-\infty}(t)\right)(z)^p \lesssim 2^{p\lambda_1\kappa} \\
&\mathbb{E}\left[\left(\delta_\kappa \overset{n}{\triangledown}_{-\infty}(t_1)\right)(z) - \left(\delta_\kappa \overset{n}{\triangledown}_{-\infty}(t_2)\right)(z)\right]^p \lesssim |t_1 - t_2|^{n\frac{p}{2}\gamma} 2^{p\lambda_2\kappa}.
\end{aligned}$$

The result then follows from [MW17c, Lemma 5.2, Lemma 5.3], the usual Kolmogorov criterion and the embedding $\mathcal{B}_{p,p}^{-\alpha+\frac{2}{p}} \hookrightarrow \mathcal{C}^{-\alpha}$, for every $\alpha > \frac{2}{p}$. □

Appendix E

Proof of Proposition 2.3. For all $n \geq 1$, using the formula

$$\mathcal{H}_n(X+Y, C) = \sum_{k=0}^{n} \binom{n}{k} X^k \mathcal{H}_{n-k}(Y, C)$$

we have

$$\overset{n}{\triangledown}{}_s^N(t) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(S_1(t-s) \upharpoonleft_{-\infty}^N(s)\right)^k \mathcal{H}_{n-k}\left(\upharpoonleft_{-\infty}^N(t), \Re_N\right).$$

Thus it suffices to prove convergence only for $\overset{n}{\triangledown}{}_{-\infty}^N(t)$, $n \geq 1$. By [Nua06, Proposition 1.1.4] for $t_1, t_2 > -\infty$ and $z_1, z_2 \in \mathbb{T}^2$

$$\mathbb{E}\overset{n}{\triangledown}{}_{-\infty}^N(t_1, z_1)\overset{n}{\triangledown}{}_{-\infty}^N(t_2, z_2) = n! \left(\mathbb{E}\upharpoonleft_{-\infty}^N(t_1, z_1)\upharpoonleft_{-\infty}^N(t_2, z_2)\right)^n.$$

For $I_m = 1 + 4\pi^2|m|^2$, $m \in \mathbb{Z}^2$, using (D.1) we get

$$\mathbb{E}\overset{n}{\triangledown}{}_{-\infty}^N(t_1, z_1)\overset{n}{\triangledown}{}_{-\infty}^N(t_2, z_2) = n! \sum_{\substack{|m_i| \leq N \\ i=1,2,\ldots,n \\ m=m_1+\ldots+m_n}} \prod_{i=1}^{n} \frac{\mathrm{e}^{-I_{m_i}|t_1-t_2|}}{2I_{m_i}} e_m(z_1 - z_2),$$

and by a change of variables this implies that for $\kappa \geq -1$

$$\begin{aligned}
&\mathbb{E}\left(\delta_\kappa \overset{n}{\triangledown}{}_{-\infty}^N(t_1)\right)(z_1)\left(\delta_\kappa \overset{n}{\triangledown}{}_{-\infty}^N(t_2)\right)(z_2) \\
&\quad \approx n! \sum_{m_1 \in \mathcal{A}_{2^\kappa}} \sum_{\substack{|m_i| \leq N \\ i=2,\ldots,n}} \prod_{i=1}^{n} \frac{\mathrm{e}^{-I_{m_i - m_{i-1}}|t_1-t_2|}}{2I_{m_i - m_{i-1}}} e_{m_1}(z_1 - z_2).
\end{aligned} \tag{E.1}$$

In a similar way

$$\begin{aligned}
&\mathbb{E}\left(\delta_\kappa \overset{n}{\vee}_{-\infty}(t_1)\right)(z_1)\left(\delta_\kappa \overset{n}{\vee}^{N}_{-\infty}(t_2)\right)(z_2)\\
&\quad\approx n! \sum_{m_1\in\mathcal{A}_{2^\kappa}} \sum_{\substack{|m_i|\le N\\ i=2,\dots,n}} \prod_{i=1}^{n} \frac{\mathrm{e}^{-I_{m_i-m_{i-1}}|t_1-t_2|}}{2I_{m_i-m_{i-1}}}\, e_{m_1}(z_1-z_2)
\end{aligned}\tag{E.2}$$

and for $K^\gamma(m) = \frac{1}{(1+|m|^2)^{1-\gamma}}$ combining (E.1) and (E.2) for $z_1 = z_2 = z$ and $t_1 = t_2 = t$ we have that

$$\mathbb{E}\left[\left(\delta_\kappa \overset{n}{\vee}_{-\infty}(t)\right)(z) - \left(\delta_\kappa \overset{n}{\vee}^{N}_{-\infty}(t)\right)(z)\right]^2 \lesssim \sum_{m\in\mathcal{A}_{2^\kappa}} K^0 \star^n_{>N} K^0(m),$$

while for $t_1 \neq t_2$ and every $\gamma \in (0,1)$

$$\begin{aligned}
&\mathbb{E}\Big(\left[\left(\delta_\kappa \overset{n}{\vee}_{-\infty}(t_1)\right)(z) - \left(\delta_\kappa \overset{n}{\vee}^{N}_{-\infty}(t_1)\right)(z)\right]\\
&\quad \times \left[\left(\delta_\kappa \overset{n}{\vee}_{-\infty}(t_2)\right)(z) - \left(\delta_\kappa \overset{n}{\vee}^{N}_{-\infty}(t_2)\right)(z)\right] \Big)\\
&\quad \lesssim |t_1-t_2|^{n\gamma} \sum_{m\in\mathcal{A}_{2^\kappa}} K^\gamma \star^n_{>N} K^\gamma(m).
\end{aligned}$$

Proceeding as in the proof of Theorem 2.1 (see Appendix D) and using Corollary C.3 we obtain that

$$\mathbb{E}\left[\left(\delta_\kappa \overset{n}{\vee}_{-\infty}(t)\right)(z) - \left(\delta_\kappa \overset{n}{\vee}^{N}_{-\infty}(t)\right)(z)\right]^2 \lesssim 2^{2\lambda_1\kappa} \frac{1}{(1+N^2)^{\lambda_1/2}},$$

for every $\lambda_1 \in (0,1)$, and

$$\begin{aligned}
&\mathbb{E}\Big(\Big[\left(\delta_\kappa \overset{n}{\vee}_{-\infty}(t_1)\right)(z) - \left(\delta_\kappa \overset{n}{\vee}^{N}_{-\infty}(t_1)\right)(z)\Big]\\
&\quad \times \left[\left(\delta_\kappa \overset{n}{\vee}_{-\infty}(t_2)\right)(z) - \left(\delta_\kappa \overset{n}{\vee}^{N}_{-\infty}(t_2)\right)(z)\right] \Big)\\
&\quad \lesssim |t_1-t_2|^{n\gamma} 2^{2\lambda_2\kappa} \frac{1}{(1+N^2)^{\lambda_2-n\gamma}},
\end{aligned}$$

for every $\gamma \in (0,\frac{1}{n})$ and $\lambda_2 > n\gamma$. The result then follows by Nelson's estimate (B.3) combined with the usual Kolmogorov criterion and the embedding $\mathcal{B}^{-\alpha+\frac{2}{p}}_{p,p} \hookrightarrow \mathcal{C}^{-\alpha}$, for every $\alpha > \frac{2}{p}$. $\square$

Appendix F

Lemma F.1 (Generalised Gronwall lemma)**.** *Let* $f : [0,T] \to \mathbb{R}$ *be a measurable function and* $\sigma_1 + \sigma_2 < 1$ *such that*

$$f(t) \le \mathrm{e}^{-c_0 t} a + b \int_0^t \mathrm{e}^{-c_0(t-s)} (t-s)^{-\sigma_1} s^{-\sigma_2} f(s)\,\mathrm{d}s.$$

Then there exists $c, C > 0$ *such that*

$$f(t) \le C \exp\left\{ -c_0 t + c b^{\frac{1}{1-\sigma_1-\sigma_2}} t \right\} a.$$

Proof. The lemma is essentially [HW13, Lemma 5.7] if we set $x(t) = \mathrm{e}^{c_0 t} f(t)$ with their notation. □

Lemma F.2. *Let* $\alpha + \beta < 1$ *and* $c > 0$*. Then*

$$\sup_{t\ge 0} \int_0^t (t-s)^{-\alpha} (s \wedge 1)^{-\beta} \mathrm{e}^{-c(t-s)}\,\mathrm{d}s < \infty.$$

Proof. Assume $t \ge 1$. Then

$$\int_0^1 (t-s)^{-\alpha} (s \wedge 1)^{-\beta} \mathrm{e}^{-c(t-s)}\,\mathrm{d}s \lesssim \mathrm{e}^{-ct} \int_0^t (t-s)^{-\alpha} (s \wedge 1)^{-\beta}\,\mathrm{d}s \lesssim t^{1-\alpha-\beta} \mathrm{e}^{-ct}$$

and

$$\begin{aligned} \int_1^t (t-s)^{-\alpha} (s \wedge 1)^{-\beta} \mathrm{e}^{-c(t-s)}\,\mathrm{d}s &\le \int_0^t s^{-\alpha} \mathrm{e}^{-cs}\,\mathrm{d}s \lesssim 1 + \int_1^t s^{-\alpha} \mathrm{e}^{-cs}\,\mathrm{d}s \\ &\lesssim 1 + \int_1^t \mathrm{e}^{-cs}\,\mathrm{d}s. \end{aligned}$$

The above implies that

$$\sup_{t\ge 1} \int_0^t (t-s)^{-\alpha} (s \wedge 1)^{-\beta} \mathrm{e}^{-c(t-s)}\,\mathrm{d}s < \infty.$$

The bound for $t \le 1$ follows easily. □

Appendix G

In this appendix we discuss some useful results about the deterministic system (5.2). Propositions G.1 and G.2 are a consequence of [FJL82, Section 8] and [KORVE07, Appendix B.1]. Although the results in [FJL82, Section 8] concern

1 space-dimension they can be easily generalised in 2 space-dimensions. For consistency we have also replaced the space H^1 appearing in [FJL82, Section 8] by $\mathcal{B}^1_{2,2}$. The fact that these spaces coincide is immediate from Definition 1.12 for $p = q = 2$ if we rewrite $\|f * \eta_k\|_{L^2}$ using Plancherel's identity.

Proposition G.1. *For every $x \in \mathcal{B}^1_{2,2}$ there exists $x_* \in \{-1, 0, 1\}$ such that $X_{det}(t; x) \overset{\mathcal{B}^1_{2,2}}{\to} x_*$.*

Proposition G.2. *For every $\delta > 0$ there exists $x_\pm \in B_{\mathcal{B}^1_{2,2}}(0; \delta)$ such that*

$$X_{det}(t; x_\pm) \overset{\mathcal{B}^1_{2,2}}{\to} \pm 1.$$

Proposition G.3. *Let $R > 0$. Then there exists $C \equiv C(R) > 0$ such that for every $\lambda > 0$ sufficiently small*

$$\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}} \le R} \|X_{det}(1; x)\|_{\mathcal{C}^{2+\lambda}} \le C.$$

Proof. By Theorems 3.6 and 3.12 there exists $C \equiv C(R) > 0$ such that

$$\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}} \le R} \sup_{t \le 1} t^\gamma \|X_{\mathrm{det}}(t; x)\|_{\mathcal{C}^\beta} \le C.$$

Let $S(t) = \mathrm{e}^{\Delta t}$. Using the mild form we write

$$X_{det}(1; x) = S(1/2) X_{det}(1/2; x) - \int_{1/2}^1 S(1-s) \left(X_{det}(s; x)^3 + X_{det}(s; x) \right) \mathrm{d}s.$$

Then

$$\begin{aligned} &\|X_{det}(1; x)\|_{\mathcal{C}^{2+\lambda}} \\ &\quad \lesssim \|X_{det}(1/2; x)\|_{\mathcal{C}^\beta} + \int_{1/2}^1 (1-s)^{-\frac{2+\lambda-\beta}{2}} \left(\|X_{det}(s; x)\|^3_{\mathcal{C}^\beta} + \|X_{det}(s; x)\|_{\mathcal{C}^\beta} \right) \end{aligned}$$

and if we choose $\lambda < \beta$ the above implies that

$$\begin{aligned} &\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}} \le R} \|X_{det}(1; x)\|_{\mathcal{C}^{2+\lambda}} \\ &\quad \lesssim \sup_{\|x\|_{\mathcal{C}^{-\alpha_0}} \le R} \sup_{t \le 1} t^{3\gamma} \|X_{\mathrm{det}}(t; x)\|^3_{\mathcal{C}^\beta} + \sup_{\|x\|_{\mathcal{C}^{-\alpha_0}} \le R} \sup_{t \le 1} t^\gamma \|X_{\mathrm{det}}(t; x)\|_{\mathcal{C}^\beta}. \end{aligned}$$

□

Appendix H

Proposition H.1. *For every $n \geq 1$ there exists $c \equiv c(n) > 0$ such that*

$$\sup_{k\geq 0} \mathbb{E} \exp \left\{ c \left(\sup_{t\in[k,k+1]} (t\wedge 1)^{(n-1)\alpha'} \|\overset{n}{\triangledown}(t)\|_{\mathcal{C}^{-\alpha}} \right)^{\frac{2}{n}} \right\} < \infty.$$

Proof. Following step by step the proof of Theorem 2.1 but using the explicit bound in Nelson's estimate (B.3) (see also [Bog07, Section 1.6]), we have that for every $p \geq 1$

$$\sup_{k\geq 0} \mathbb{E} \left(\sup_{t\in[k,k+1]} (t\wedge 1)^{(n-1)\alpha'} \|\overset{n}{\triangledown}(t)\|_{\mathcal{C}^{-\alpha}} \right)^{p} \leq (p-1)^{\frac{n}{2}p} C_n^{\frac{p}{2}},$$

for some $C_n > 0$. Then for any $c > 0$

$$\begin{aligned}
&\mathbb{E} \exp \left\{ c \left(\sup_{t\in[k,k+1]} (t\wedge 1)^{(n-1)\alpha'} \|\overset{n}{\triangledown}(t)\|_{\mathcal{C}^{-\alpha}} \right)^{\frac{2}{n}} \right\} \\
&\quad = \sum_{k\geq 0} \frac{c^p \mathbb{E}\left(\sup_{t\in[k,k+1]} (t\wedge 1)^{(n-1)\alpha'} \|\overset{n}{\triangledown}_{-\infty}(t)\|_{\mathcal{C}^{-\alpha}}\right)^{\frac{2}{n}p}}{p!} \\
&\quad \leq \sum_{p\geq 0} \frac{c^p (p-1)^p (C_n)^{\frac{p}{n}}}{p!}
\end{aligned}$$

and by choosing $c \equiv c(n) > 0$ sufficiently small the series converges. □

Appendix I

Lemma I.1. *Let $g_1, \tilde{g}_1$ be positive random variables such that*

$$\mathbb{P}(g_1 \geq g) \leq \mathbb{P}(\tilde{g}_1 \geq g)$$

for every $g \geq 0$ and let F be a positive decreasing measurable function on $[0, \infty)$. Then

$$\int_0^\infty F(g)\, \mu_{g_1}(\mathrm{d}g) \geq \int_0^\infty F(g)\, \mu_{\tilde{g}_1}(\mathrm{d}g)$$

where μ_{g_1} and $\mu_{\tilde{g}_1}$ is the law of g_1 and $\tilde{g}_1$.

Proof. We first assume that F is smooth. Then $\frac{\mathrm{d}}{\mathrm{d}g}F(g) \le 0$ for every $g \ge 0$. Hence

$$\begin{aligned}\int_0^\infty F(g)\,\mu_{g_1}(\,\mathrm{d}g) &= F(0) + \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}g}F(g)\,\mathbb{P}(g_1 \ge g)\,\mathrm{d}g \\ &\ge F(0) + \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}g}F(g)\,\mathbb{P}(\tilde{g}_1 \ge g)\,\mathrm{d}g \\ &= \int_0^\infty F(g)\,\mu_{\tilde{g}_1}(\,\mathrm{d}g)\end{aligned}$$

which proves the estimate for F differentiable. To prove the estimate for a general decreasing function F we define $F_\delta = F * \eta_\delta$ for some positive mollifier η_δ to preserve monotonicity and use the last estimate together with the dominated convergence theorem. $\square$

Appendix J

Proof of Proposition 5.32. By Proposition 2.3 for every $\alpha > 0$, $p \ge 1$ and $T > 0$

$$\lim_{N\to\infty} \mathbb{E}\left(\sup_{t\le T}(t\wedge 1)^{(n-1)\alpha'}\|\vee_n^N(t) - \vee_n(t)\|_{\mathcal{C}^{-\alpha}}\right)^p = 0.$$

Hence $\sup_{t\le T}(t\wedge 1)^{(n-1)\alpha'}\|\vee_n^N(t) - \vee_n(t)\|_{\mathcal{C}^{-\alpha}}$ convergences to 0 in probability.

It is enough to prove that

$$\lim_{N\to\infty}\sup_{x\in\aleph}\sup_{t\le T}(t\wedge 1)^{\gamma}\|v^N(t;x_N) - v(t;x)\|_{\mathcal{C}^\beta} = 0.$$

This, convergence in probability of $\sup_{t\le T}\|\uparrow^N(t) - \uparrow(t)\|_{\mathcal{C}^{-\alpha}}$ to 0 and the embedding $\mathcal{C}^\beta \subset \mathcal{C}^{-\alpha}$ (see (A.1)) imply the result.

Let $S(t) = \mathrm{e}^{\Delta t}$. For simplicity we write $v^N(t)$ and $v(t)$ to denote $v^N(t;x_N)$ and $v(t;x)$. Using the mild forms of (5.49) and (5.6) we get

$$\|v^N(t) - v(t)\|_{\mathcal{C}^\beta} \le \sum_{i=1}^{7} I_i \tag{J.1}$$

where

$$I_1 := \|S(t)(x_N - x)\|_{\mathcal{C}^\beta},\ I_2 := \int_0^t \|S(t-s)[\Pi_N(v^N(s)^3) - v(s)^3]\|_{\mathcal{C}^\beta}\,\mathrm{d}s$$

$$I_3 := 3\int_0^t \|S(t-s)\left[\Pi_N\left(v^N(s)^2\varepsilon^{\frac{1}{2}}\uparrow^N(s)\right) - v(s)^2\varepsilon^{\frac{1}{2}}\uparrow(s)\right]\|_{\mathcal{C}^\beta}\,\mathrm{d}s$$

$$I_4 := 3\int_0^t \|S(t-s)[\Pi_N(v^N(s)\varepsilon\vee^N(s)) - v(s)\varepsilon\vee(s)]\|_{\mathcal{C}^\beta}\,\mathrm{d}s$$

$$I_5 := \int_0^t \|S(t-s)\left(\Pi_N \varepsilon^{\frac{3}{2}} \Psi^N(s) - \varepsilon^{\frac{3}{2}} \Psi(s)\right)\|_{\mathcal{C}^\beta}\,\mathrm{d}s$$

$$I_6 := 2\int_0^t \|S(t-s)\left(\varepsilon^{\frac{1}{2}} \uparrow^N(s) - \varepsilon^{\frac{1}{2}} \uparrow(s)\right)\|_{\mathcal{C}^\beta}\,\mathrm{d}s$$

$$I_7 := \int_0^t \|S(t-s)(v^N(s) - v(s))\|_{\mathcal{C}^\beta}\,\mathrm{d}s.$$

Let $\iota = \inf\{t > 0 : (t\wedge 1)^\gamma \|v^N(t) - v(t)\|_{\mathcal{C}^\beta} \geq 1\}$ and $t \leq T \wedge \iota$. We treat each of the terms in (J.1) separately. Below the parameters α and λ can be taken arbitrarily small and all the implicit constants depend on $\sup_{t\leq T}(t\wedge 1)^{(n-1)\alpha'}\|\bigvee^{n}(t)\|_{\mathcal{C}^{-\alpha}}$, and $\sup_{x\in\aleph}\sup_{t\leq T}(t\wedge 1)^\gamma\|v(t)\|_{\mathcal{C}^\beta}$.

Term I_1:

$$I_1 \overset{(A.7)}{\lesssim} (t\wedge 1)^{-\frac{\alpha_0+\beta}{2}} \sup_{x\in\aleph}\|x_N - x\|_{\mathcal{C}^{-\alpha_0}}$$

Term I_2:

$$\begin{aligned}
I_2 &\overset{(A.7)}{\lesssim} \int_0^t \left((t-s)^{-\frac{\lambda}{2}}\|\Pi_N(v^N(s)^3) - v^N(s)^3\|_{\mathcal{C}^{\beta-\lambda}} + \|v^N(s)^3 - v(s)^3\|_{\mathcal{C}^\beta}\right)\mathrm{d}s \\
&\overset{(A.15)}{\lesssim} \int_0^t (t-s)^{-\frac{\lambda}{2}}\left(\frac{(\log N)^2}{N^\lambda}\|v^N(s)^3\|_{\mathcal{C}^\beta} + \|v^N(s)^3 - v(s)^3\|_{\mathcal{C}^{\beta-\lambda}}\right)\mathrm{d}s \\
&\overset{(A.10)}{\lesssim} \int_0^t \Bigg[(t-s)^{-\frac{\lambda}{2}}\frac{(\log N)^2}{N^\lambda}\|v^N(s)\|^3_{\mathcal{C}^\beta} + \|v^N(s) - v(s)\|_{\mathcal{C}^\beta} \\
&\qquad \times\left(\|v^N(s)\|^2_{\mathcal{C}^\beta} + \|v^N(s)\|_{\mathcal{C}^\beta}\|v(s)\|_{\mathcal{C}^\beta} + \|v(s)\|^2_{\mathcal{C}^\beta}\right)\Bigg]\mathrm{d}s \\
&\lesssim \int_0^t \Bigg((t-s)^{-\frac{\beta+\frac{2}{p}-1}{2}}\frac{(\log N)^2}{N^\lambda}(s\wedge 1)^{-3\gamma} + (s\wedge 1)^{-2\gamma} \\
&\qquad \times \|v^N(s) - v(s)\|_{\mathcal{C}^\beta}\Bigg)\mathrm{d}s.
\end{aligned}$$

Term I_3:

$$\begin{aligned}
I_3 &\overset{(A.7)}{\lesssim} \int_0^t (t-s)^{-\frac{\alpha+\beta+\lambda}{2}}\|\Pi_N(v^N(s)^2\uparrow^N(s)) - v(s)^2\uparrow(s)\|_{\mathcal{C}^{-\alpha-\lambda}}\,\mathrm{d}s \\
&\overset{(A.15)}{\lesssim} \int_0^t (t-s)^{-\frac{\alpha+\beta+\lambda}{2}}\Bigg(\frac{(\log N)^2}{N^\lambda}\|v^N(s)^2\uparrow^N(s)\|_{\mathcal{C}^{-\alpha}} \\
&\qquad + \|v^N(s)^2(\uparrow^N(s) - \uparrow(s))\|_{\mathcal{C}^{-\alpha}} + \|\uparrow(s)(v^N(s)^2 - v(s)^2)\|_{\mathcal{C}^{-\alpha}}\Bigg)\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\overset{(A.11),(A.10)}{\lesssim} \int_0^t (t-s)^{-\frac{\alpha+\beta+\lambda}{2}} \bigg(\frac{(\log N)^2}{N^\lambda} \|v^N(s)\|^2_{\mathcal{C}^\beta} \|\Psi^N(s)\|_{\mathcal{C}^{-\alpha}} + \|v^N(s)\|^2_{\mathcal{C}^\beta} \\
&\quad \times \|\Psi^N(s) - \Psi(s)\|_{\mathcal{C}^{-\alpha}} + \left(\|v^N(s)\|_{\mathcal{C}^\beta} + \|v(s)\|_{\mathcal{C}^\beta}\right) \|v^N(s) - v(s)\|_{\mathcal{C}^\beta} \\
&\quad \times \|\Psi(s)\|_{\mathcal{C}^{-\alpha}} \bigg) \,\mathrm{d}s \\
&\lesssim \int_0^t (t-s)^{-\frac{\alpha+\beta+\lambda}{2}} \bigg(\frac{(\log N)^2}{N^\lambda} (s\wedge 1)^{-2\gamma} + (s \wedge 1)^{-2\gamma} \|\Psi^N(s) - \Psi(s)\|_{\mathcal{C}^{-\alpha}} \\
&\quad + (s\wedge 1)^{-\gamma} \|v^N(s) - v(s)\|_{\mathcal{C}^\beta} \bigg) \,\mathrm{d}s.
\end{aligned}
$$

Term I_4: Similarly to I_3,

$$
\begin{aligned}
I_4 &\lesssim \int_0^t (t-s)^{-\frac{\alpha+\beta+\lambda}{2}} \bigg(\frac{(\log N)^2}{N^\lambda} (s\wedge 1)^{-\gamma-\alpha'} + (s\wedge 1)^{-\gamma} \|\Psi_2^N(s) - \Psi_2(s)\|_{\mathcal{C}^{-\alpha}} \\
&\quad + (s\wedge 1)^{-\alpha'} \|v^N(s) - v(s)\|_{\mathcal{C}^\beta} \bigg) \,\mathrm{d}s.
\end{aligned}
$$

Term I_5:

$$
\begin{aligned}
I_5 &\overset{(A.7)}{\lesssim} \int_0^t (t-s)^{-\frac{\alpha+\beta+\lambda}{2}} \|\Pi_N \Psi_3^N(s) - \Psi_3(s)\|_{\mathcal{C}^{-\alpha-\lambda}} \,\mathrm{d}s \\
&\overset{(A.15)}{\lesssim} \int_0^t (t-s)^{-\frac{\alpha+\beta+\lambda}{2}} \left(\frac{(\log N)^2}{N^\lambda} (s\wedge 1)^{-2\alpha'} + \|\Psi_3^N(s) - \Psi_3(s)\|_{\mathcal{C}^{-\alpha}} \right) \,\mathrm{d}s.
\end{aligned}
$$

Terms I_6, I_7:

$$
\begin{aligned}
I_6 &\overset{(A.7)}{\lesssim} \int_0^t (t-s)^{-\frac{\alpha+\beta}{2}} \|\Psi^N(s) - \Psi(s)\|_{\mathcal{C}^{-\alpha}} \,\mathrm{d}s. \\
I_7 &\overset{(A.7)}{\lesssim} \int_0^t \|v^N(s) - v(s)\|_{\mathcal{C}^\beta} \,\mathrm{d}s.
\end{aligned}
$$

Combining the above estimates we obtain that for $t \leq T \wedge \iota$

$$
\begin{aligned}
&\|v^N(t) - v(t)\|_{\mathcal{C}^\beta} \\
&\quad \lesssim (t\wedge 1)^{-\frac{\alpha_0+\beta}{2}} \sup_{x\in\aleph} \|x_N - x\|_{\mathcal{C}^{-\alpha_0}} \\
&\qquad + T^{1-\frac{\alpha+\beta+\lambda}{2}-3\gamma} \left(\frac{(\log N)^2}{N} + \sup_{t\leq T} (t\wedge 1)^{(n-1)\alpha'} \|\Psi_n^N(t) - \Psi_n(t)\|_{\mathcal{C}^{-\alpha}} \right) \\
&\qquad + \int_0^t (t-s)^{-\frac{\alpha+\beta+\lambda}{2}} (s\wedge 1)^{-2\gamma} \|v^N(s) - v(s)\|_{\mathcal{C}^\beta} \,\mathrm{d}s.
\end{aligned}
$$

By Lemma F.1 on $f(t) = (t \wedge 1)^\gamma \|v^N(t) - v(t)\|_{\mathcal{C}^\beta}$ we find $C \equiv C(T) > 0$ such that

$$\sup_{t \le T \wedge \iota} (t \wedge 1)^\gamma \|v^N(t) - v(t)\|_{\mathcal{C}^\beta} \le C \Bigg(\sup_{x \in \aleph} \|x_N - x\|_{\mathcal{C}^{-\alpha_0}} + \frac{(\log N)^2}{N} \\ + \sup_{t \le T} (t \wedge 1)^{(n-1)\alpha'} \|\overset{n}{\triangledown}^N(t) - \overset{n}{\triangledown}(t)\|_{\mathcal{C}^{-\alpha}} \Bigg).$$

This and convergence of $\sup_{t \le T} \|\overset{n}{\triangledown}^N(t) - \overset{n}{\triangledown}(t)\|_{\mathcal{C}^{-\alpha}}$ to 0 in probability imply the result. □

Appendix K

In this section we fix $\beta \in \left(\frac{1}{3}, \frac{2}{3}\right)$, $\gamma \in \left(\frac{\beta}{2}, \frac{1}{3}\right)$ and $p \in (1, 2)$ such that

$$1 - \frac{2}{3p} < \beta \text{ and } 1 - \frac{\beta + \frac{2}{p} - 1}{2} - 2\gamma > 0.$$

The next proposition provides local existence of (5.49) in $\mathcal{B}^\beta_{2,2}$ up to some time $T_* > 0$ which is uniform in the regularisation parameter N.

Proposition K.1. *Let $K, R, T > 0$ such that $\|x\|_{\mathcal{B}^{-\alpha_0}_{2,2}} \le R$ and*

$$\sup_{t \le T} (t \wedge 1)^{(n-1)\alpha'} \|\overset{n}{\triangledown}^N(t)\|_{\mathcal{B}^{-\alpha}_{\infty,2}} \le K.$$

Then there exist $T_* \equiv T_*(K, R) \le T$ and $C \equiv C(K, R) > 0$ such that (5.49) *has a unique solution $v \in C((0, T_*]; \mathcal{B}^\beta_{2,2})$ satisfying*

$$\sup_{t \le T_*} (t \wedge 1)^\gamma \|v^N(t; x)\|_{\mathcal{B}^\beta_{2,2}} \le C.$$

Proof. Let $S(t) = \mathrm{e}^{\Delta t}$. We define

$$\mathscr{T}(v)(t) := S(t)x - \int_0^t S(t-s) \Pi_N \Big(v(s)^3 + 3v(s)^2 \varepsilon^{\frac{1}{2}} \uparrow^N(s) + 3v(s) \varepsilon \vee^N(s) \\ + \varepsilon^{\frac{3}{2}} \Psi^N(s) \Big) \,\mathrm{d}s + 2 \int_0^t S(t-s) \left(\varepsilon^{\frac{1}{2}} \uparrow^N(s) + v(s) \right) \mathrm{d}s.$$

It is enough to prove that there exists $T_* > 0$ such that $\mathscr{T}$ is a contraction on

$$\mathscr{B}_{T_*} := \left\{ v : \sup_{t \le T_*} (t \wedge 1)^\gamma \|v(t; x)\|_{\mathcal{B}^\beta_{2,2}} \le 1 \right\}.$$

We first prove that for $T_* > 0$ sufficiently small $\mathscr{T}$ maps $\mathscr{B}_{T_*}$ to itself. To do so we notice that

$$\|\mathscr{T}(v)(t)\|_{\mathcal{B}^{\beta}_{2,2}} \lesssim \sum_{i=1}^{7} I_i$$

where

$$\begin{aligned}
&I_1 := \|S(t)x\|_{\mathcal{B}^{\beta}_{2,2}},\ I_2 := \int_0^t \|S(t-s)v(s)^3\|_{\mathcal{B}^{\beta}_{2,2}}\,\mathrm{d}s,\\
&I_3 := \int_0^t \|S(t-s)(v(s)^2\mathbf{I}^N(s))\|_{\mathcal{B}^{\beta}_{2,2}}\,\mathrm{d}s,\\
&I_4 := \int_0^t \|S(t-s)(v(s)\mathbf{V}^N(s))\|_{\mathcal{B}^{\beta}_{2,2}}\,\mathrm{d}s,\ I_5 := \int_0^t \|S(t-s)\mathbf{\Psi}^N(s)\|_{\mathcal{B}^{\beta}_{2,2}}\,\mathrm{d}s,\\
&I_6 := \int_0^t \|S(t-s)\mathbf{I}^N(s)\|_{\mathcal{B}^{\beta}_{2,2}}\,\mathrm{d}s,\ I_7 := \int_0^t \|S(t-s)v(s)\|_{\mathcal{B}^{\beta}_{2,2}}\,\mathrm{d}s.
\end{aligned}$$

Here we use (A.18) together with the relation $S(\cdot)\Pi_N = \Pi_N S(\cdot)$ to drop Π_N. We treat each term separately.

<u>Term I_1:</u>

$$I_1 \overset{\text{(A.7)}}{\lesssim} (t\wedge 1)^{-\frac{\alpha_0+\beta}{2}}\|x\|_{\mathcal{B}^{-\alpha_0}_{2,2}} \lesssim (t\wedge 1)^{-\frac{\alpha_0+\beta}{2}} R.$$

<u>Term I_2:</u>

$$\begin{aligned}
I_2 &\overset{\text{(A.6)}}{\lesssim} \int_0^t \|S(t-s)v(s)^3\|_{\mathcal{B}^{\beta+\frac{2}{p}-1}_{p,2}}\,\mathrm{d}s \overset{\text{(A.7)}}{\lesssim} \int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1}{2}}\|v(s)^3\|_{\mathcal{B}^{0}_{p,2}}\,\mathrm{d}s\\
&\overset{\text{(A.10)}}{\lesssim} \int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1}{2}}\|v(s)\|^3_{\mathcal{B}^{0}_{3p,2}}\,\mathrm{d}s \overset{\text{(A.6)}}{\lesssim} \int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1}{2}}\|v(s)\|^3_{\mathcal{B}^{1-\frac{2}{3p}}_{2,2}}\,\mathrm{d}s\\
&\overset{1-\frac{2}{3p}<\beta}{\lesssim} \int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1}{2}}\|v(s)\|^3_{\mathcal{B}^{\beta}_{2,2}}\,\mathrm{d}s \lesssim \int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1}{2}}(s\wedge 1)^{-3\gamma}\,\mathrm{d}s.
\end{aligned}$$

<u>Term I_3:</u>

$$\begin{aligned}
I_3 &\overset{\text{(A.6),(A.7)}}{\lesssim} \int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1+\alpha}{2}}\|v(s)^2\mathbf{I}^N(s)\|_{\mathcal{B}^{-\alpha}_{p,2}}\,\mathrm{d}s\\
&\overset{\text{(A.11),(A.10)}}{\lesssim} K\int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1+\alpha}{2}}\|v(s)\|^2_{\mathcal{B}^{\alpha+\lambda}_{2p,2}}\,\mathrm{d}s\\
&\overset{\text{(A.6)}}{\lesssim} K\int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1+\alpha}{2}}\|v(s)\|^2_{\mathcal{B}^{\alpha+\lambda+1-\frac{1}{p}}_{2,2}}\,\mathrm{d}s
\end{aligned}$$

$$\overset{1-\frac{2}{3p}<\beta}{\lesssim} K \int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1+\alpha}{2}} \|v(s)\|^2_{\mathcal{B}^{\beta}_{2,2}} \,\mathrm{d}s$$
$$\lesssim K \int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1+\alpha}{2}} (s \wedge 1)^{-2\gamma} \,\mathrm{d}s.$$

Term I_4:

$$I_4 \overset{\text{(A.6),(A.7)}}{\lesssim} \int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1+\alpha}{2}} \|v(s)\mathbf{V}^N(s)\|_{\mathcal{B}^{-\alpha}_{p,2}} \,\mathrm{d}s$$
$$\overset{\text{(A.11),(A.6)}}{\lesssim} K \int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1+\alpha}{2}} (s \wedge 1)^{-\alpha'} \|v(s)\|_{\mathcal{B}^{\alpha+\lambda+1-\frac{2}{p}}_{2,2}} \,\mathrm{d}s$$
$$\overset{1-\frac{2}{3p}<\beta}{\lesssim} K \int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1+\alpha}{2}} (s \wedge 1)^{-\alpha'} \|v(s)\|_{\mathcal{B}^{\beta}_{2,2}} \,\mathrm{d}s$$
$$\lesssim K \int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1+\alpha}{2}} (s \wedge 1)^{-\gamma-\alpha'} \,\mathrm{d}s.$$

Terms I_5, I_6, I_7:

$$I_5 \overset{\text{(A.7)}}{\lesssim} \int_0^t (t-s)^{-\frac{\beta+\alpha}{2}} \|\mathbf{V}^N(s)\|_{\mathcal{B}^{-\alpha}_{2,2}} \,\mathrm{d}s \lesssim K \int_0^t (t-s)^{-\frac{\beta+\alpha}{2}} (s \wedge 1)^{-2\alpha'} \,\mathrm{d}s.$$
$$I_6 \overset{\text{(A.7)}}{\lesssim} \int_0^t (t-s)^{-\frac{\beta+\alpha}{2}} \|\mathbf{I}^N(s)\|_{\mathcal{B}^{-\alpha}_{2,2}} \,\mathrm{d}s \lesssim K \int_0^t (t-s)^{-\frac{\beta+\alpha}{2}} \,\mathrm{d}s.$$
$$I_7 \overset{\text{(A.7)}}{\lesssim} \int_0^t \|v(s)\|_{\mathcal{B}^{\beta}_{2,2}} \,\mathrm{d}s \lesssim \int_0^t (s \wedge 1)^{-\gamma} \,\mathrm{d}s.$$

Combining all the above we find $C \equiv C(K,R) > 0$ such that

$$\sup_{t \le T_*} (t \wedge 1)^{\gamma} \|\mathscr{T}(v)(t)\|_{\mathcal{B}^{\beta}_{2,2}} \le C T_*^{\theta}$$

for some $\theta \equiv \theta(\alpha, \alpha', \alpha_0, \beta, \gamma) \in (0,1)$. Choosing $T_* > 0$ sufficiently small the above implies that

$$\sup_{t \le T_*} (t \wedge 1)^{\gamma} \|\mathscr{T}(v)(t)\|_{\mathcal{B}^{\beta}_{2,2}} \le 1.$$

Hence for this choice of T_*, $\mathscr{T}$ maps $\mathscr{B}_{T_*}$ to itself. In a similar way, but by possibly choosing a smaller value of T_*, we prove that $\mathscr{T}$ is a contraction on $\mathscr{B}_{T_*}$. For simplicity we omit the proof. That way we obtain a unique solution $v \in C((0,T_*]; \mathcal{B}^{\beta}_{2,2})$. We can furthermore assume that T_* is maximal in the sense that either $T_* = T$ or $\lim_{t \nearrow T_*} \|v(t;x)\|_{\mathcal{B}^{\beta}_{2,2}} = \infty$. $\square$

Proposition K.2. *For every* $t_0 \in (0,1)$ *and* $K, R > 0$ *there exists* $C > 0$ *such that if* $\|x\|_{\mathcal{B}^{-\alpha}_{2,2}} \le R$ *and* $\sup_{t\le 1} t^{(n-1)\alpha'} \|\varepsilon^{\frac{n}{2}} \mathbf{\Psi}_n^N(t)\|_{\mathcal{C}^{-\alpha}} \le K$ *then*

$$\sup_{\|x\|_{\mathcal{B}^{-\alpha}_{2,2}}\le R} \|X_N(t_0;x)\|_{\mathcal{C}^{-\alpha}} \le C.$$

Proof. Using the a priori estimate in Proposition 5.33 we can assume that $T_* = 1$ in Proposition K.1. This implies that

$$\sup_{\|x\|_{\mathcal{B}^{-\alpha}_{2,2}}\le R} \sup_{t\le 1} t^{\gamma} \|v^N(t;x)\|_{\mathcal{B}^{\beta}_{2,2}} \le C. \tag{K.1}$$

For simplicity we assume that $t_0 = 1$. Let $S(t) = \mathrm{e}^{\Delta t}$. Using the mild form of (5.49) we obtain that

$$\|v^N(1)\|_{\mathcal{C}^{-\alpha}} \lesssim \sum_{i=1}^{7} I_i$$

where

$$I_1 := \|S(1/2)v^N(1/2)\|_{\mathcal{C}^{-\alpha}},\ I_2 := \int_{1/2}^{1} \|S(1-s)\Pi_N(v^N(s))^3\|_{\mathcal{C}^{-\alpha}}\,\mathrm{d}s,$$

$$I_3 := \int_{1/2}^{1} \|S(1-s)\Pi_N\left(v^N(s)^2\varepsilon^{\frac{1}{2}}\mathbf{I}^N(s)\right)\|_{\mathcal{C}^{-\alpha}}\,\mathrm{d}s,$$

$$I_4 := \int_{1/2}^{1} \|S(1-s)\Pi_N\left(v^N(s)\varepsilon\mathbf{V}^N(s)\right)\|_{\mathcal{C}^{-\alpha}}\,\mathrm{d}s,$$

$$I_5 := \int_{1/2}^{1} \|S(1-s)\Pi_N\varepsilon^{\frac{3}{2}}\mathbf{\Psi}^N(s)\|_{\mathcal{C}^{-\alpha}}\,\mathrm{d}s,\ I_6 := \int_{1/2}^{1} \|S(1-s)\varepsilon^{\frac{1}{2}}\mathbf{I}^N(s)\|_{\mathcal{C}^{-\alpha}}\,\mathrm{d}s,$$

$$I_7 := \int_{1/2}^{1} \|S(1-s)v^N(s)\|_{\mathcal{C}^{-\alpha}}\,\mathrm{d}s.$$

We treat each term separately.

Term I_1:

$$I_1 \overset{\text{(A.6)}}{\lesssim} \|S(1/2)v^N(1/2)\|_{\mathcal{B}^{-\alpha+1}_{2,\infty}} \overset{\text{(A.7)}}{\lesssim} \|v^N(1/2)\|_{\mathcal{B}^{-\alpha}_{2,\infty}} \lesssim \|v^N(1/2)\|_{\mathcal{B}^{-\alpha}_{2,2}}$$

Term I_2:

$$\begin{aligned} I_2 &\overset{\text{(A.6)}}{\lesssim} \int_{1/2}^{1} \|S(1-s)\Pi_N(v^N(s)^3)\|_{\mathcal{B}^{-\alpha+\frac{2}{p}}_{p,\infty}}\,\mathrm{d}s \\ &\overset{\text{(A.7)}}{\lesssim} \int_{1/2}^{1} (1-s)^{-\frac{-\alpha+\frac{2}{p}+\lambda}{2}} \|\Pi_N(v^N(s)^3)\|_{\mathcal{B}^{-\lambda}_{p,\infty}}\,\mathrm{d}s \\ &\overset{\text{(A.16),(A.10)}}{\lesssim} \int_{1/2}^{1} (1-s)^{-\frac{-\alpha+\frac{2}{p}+\lambda}{2}} \|v^N(s)\|^3_{\mathcal{B}^{0}_{3p,\infty}}\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&\overset{(A.6)}{\lesssim} \int_{1/2}^{1} (1-s)^{-\frac{-\alpha+\frac{2}{p}+\lambda}{2}} \|v^N(s)\|^3_{\mathcal{B}^{1-\frac{2}{3p}}_{2,\infty}} \,\mathrm{d}s \\
&\overset{1-\frac{2}{3p}<\beta}{\lesssim} \int_{1/2}^{1} (1-s)^{-\frac{-\alpha+\frac{2}{p}+\lambda}{2}} \|v^N(s)\|^3_{\mathcal{B}^{\beta}_{2,2}} \,\mathrm{d}s.
\end{aligned}
$$

Term I_3:

$$
\begin{aligned}
I_3 &\overset{(A.6)}{\lesssim} \int_{1/2}^{1} \left\|S(1-s)\Pi_N\left(v^N(s)^2\varepsilon^{\frac{1}{2}}\mathbf{I}^N(s)\right)\right\|_{\mathcal{B}^{-\alpha+\frac{2}{p}}_{p,\infty}} \,\mathrm{d}s \\
&\overset{(A.7)}{\lesssim} \int_{1/2}^{1} (1-s)^{-\frac{\frac{2}{p}+\lambda}{2}} \left\|\Pi_N\left(v^N(s)^2\varepsilon^{\frac{1}{2}}\mathbf{I}^N(s)\right)\right\|_{\mathcal{B}^{-\alpha-\lambda}_{p,\infty}} \,\mathrm{d}s \\
&\overset{(A.16),(A.11),(A.10)}{\lesssim} \int_{1/2}^{1} (1-s)^{-\frac{\frac{2}{p}+\lambda}{2}} \|v^N(s)\|^2_{\mathcal{B}^{\alpha+\lambda}_{2p,\infty}} \|\varepsilon^{\frac{1}{2}}\mathbf{I}^N(s)\|_{\mathcal{C}^{-\alpha}} \,\mathrm{d}s \\
&\overset{(A.6)}{\lesssim} \int_{1/2}^{1} (1-s)^{-\frac{\frac{2}{p}+\lambda}{2}} \|v^N(s)\|^2_{\mathcal{B}^{\alpha+\lambda+1-\frac{1}{p}}_{2,\infty}} \|\varepsilon^{\frac{1}{2}}\mathbf{I}^N(s)\|_{\mathcal{C}^{-\alpha}} \,\mathrm{d}s \\
&\overset{1-\frac{2}{3p}<\beta}{\lesssim} \int_{1/2}^{1} (1-s)^{-\frac{\frac{2}{p}+\lambda}{2}} \|v^N(s)\|^2_{\mathcal{B}^{\beta}_{2,2}} \|\varepsilon^{\frac{1}{2}}\mathbf{I}^N(s)\|_{\mathcal{C}^{-\alpha}} \,\mathrm{d}s.
\end{aligned}
$$

Term I_4: Similarly to I_3,

$$
I_4 \lesssim \int_{1/2}^{1} (1-s)^{-\frac{\frac{2}{p}+\lambda}{2}} \|v^N(s)\|_{\mathcal{B}^{\beta}_{2,2}} \|\varepsilon\mathbf{V}^N(s)\|_{\mathcal{C}^{-\alpha}} \,\mathrm{d}s.
$$

TermsI_5:

$$
\begin{aligned}
I_5 &\overset{(A.7)}{\lesssim} \int_{1/2}^{1} (1-s)^{-\frac{\lambda}{2}} \|\Pi_N\varepsilon^{\frac{3}{2}}\mathbf{\Psi}^N(s)\|_{\mathcal{C}^{-\alpha-\lambda}} \,\mathrm{d}s \\
&\overset{(A.16)}{\lesssim} \int_{1/2}^{1} (1-s)^{-\frac{\lambda}{2}} \|\varepsilon^{\frac{3}{2}}\mathbf{\Psi}^N(s)\|_{\mathcal{C}^{-\alpha-\lambda}} \,\mathrm{d}s.
\end{aligned}
$$

Terms I_6, I_7:

$$
\begin{aligned}
I_6 &\overset{(A.7)}{\lesssim} \int_{1/2}^{1} (1-s)^{-\frac{\lambda}{2}} \|\varepsilon^{\frac{1}{2}}\mathbf{I}^N(s)\|_{\mathcal{C}^{-\alpha-\lambda}} \,\mathrm{d}s. \\
I_7 &\overset{(A.6)}{\lesssim} \int_{1/2}^{1} \|S(1-s)v^N(s)\|_{\mathcal{B}^{-\alpha+1}_{2,2}} \,\mathrm{d}s \overset{(A.7)}{\lesssim} \int_{1/2}^{1} (1-s)^{-\frac{-\alpha+1-\beta}{2}} \|v^N(s)\|_{\mathcal{B}^{\beta}_{2,2}} \,\mathrm{d}s.
\end{aligned}
$$

The proof is complete if we combine these estimates with (K.1). $\square$

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